STAT 510
Solutions to Midterm 1

1. (a) \( A \cap B \cap C \) is the event that a U.S. birth results in identical twins.
\[
P(A \cap B \cap C) = \frac{1}{90} \times \frac{1}{3} \times \frac{1}{2}
= \frac{1}{540}
\]
(5 marks)

(b) \( P(\text{No two of the pairs celebrate birthday in the same month}) = \frac{12!}{12^{12}} \)
(5 marks)

2. (a) \( P(\text{Get a pair, when picked without replacement}) = \frac{\binom{3}{1}}{\binom{6}{2}} = 0.2 \)
(5 marks)

(b) \( P(\text{Get a pair, when one right and one left shoe is picked}) = \frac{3}{3 \times 3} = \frac{1}{3} \)
(5 marks)

3. Let \( A \) be the event that the 1st ace is the 15th card to appear.
   (a) Let \( B \) be the event that the 16th card is ace of hearts.
\[
P(B|A) = \frac{\text{P(16th card is ace|A)}P(B|16th card is ace, A)}{P(A)}
= \frac{3 \times 1}{37 \times 4}
= \frac{3}{148}
\]
(5 marks)
(b) Let $C$ be the event that the 16th card is the seven of diamonds and $D$ be the event that the seven of diamonds appeared in the first 15 cards.

\[
P(C | A) = \frac{P(C | D^c, A)P(D^c | A)}{P(D^c | A)} = \frac{\frac{1}{37}(1 - \frac{14}{48})}{\frac{17}{888}}\]

(5 marks)

4. (a) \[
P(\text{at least one out of 3 approves}) = 1 - 0.4^3\]

(2 marks)

\[
P(\text{resolution will pass}) = 0.6 \times (1 - 0.4^3) + 0.4 \times 0.6^3\]

(3 marks)

(b) Let $A$ be the event that the CEO approves and $B$ be the event that the resolution is passed.

\[
P(\text{CEO approves | resolution pass}) = \frac{P(B \cap A)}{P(B \cap A) + P(B \cap A^c)P(A^c)} = \frac{0.6 \times (1 - 0.4)^3}{0.6 \times (1 - 0.4)^3 + 0.4 \times 0.6^3}\]

(3 marks)

5. (a) \[
P(X = 7) = 0.5 \left(\frac{10}{7}\right)0.4^70.6^3 + 0.5 \left(\frac{10}{7}\right)0.7^70.3^3\]

(5 marks)

(b) \[
P(\text{coin 1 is chosen | exactly 7 of 10 flips}) = \frac{P(\text{coin 1 is chosen \cap exactly 7 of 10 flips})}{P(\text{exactly 7 of 10 flips})} = \frac{0.5 \left(\frac{10}{7}\right)0.4^70.6^3}{0.5 \left(\frac{10}{7}\right)0.4^70.6^3 + 0.5 \left(\frac{10}{7}\right)0.7^70.3^3}\]

(2 + 3 = 5 marks)

6. (a) \[
P(\text{royal straight flush}) = \frac{4}{\binom{52}{5}}\]

(5 marks)

(b) When $p$ is small and $n$ is large, we can use the Poisson distribution to approximate the Binomial.

\[
n = 100 \times 52 \times 20 = 104000\]
\[
np = 104000 \times 1.6 \times 10^{-6} = 0.1664\]

\[
P(\text{never see a royal straight flush}) = P(X = 0) = e^{-0.1664}\]
\[
P(\text{see exactly 2 royal straight flushes}) = P(X = 2) = e^{-0.1664}(0.1664)^2\]

(5 marks)
7. (a) 

\[
E[X^2] = \lambda E[(X + 1)] = \lambda^2 + \lambda \\
E[X^3] = \lambda E[(X + 1)^2] = \lambda^3 + 3\lambda^2 + \lambda
\]

\((2 + 3 = 5 \text{ marks})\)

(b) 

\[
E[X^n] = \sum_{x=0}^{\infty} x^n \frac{e^{-\lambda} \lambda^x}{x!} \\
= \sum_{x=1}^{\infty} x^n \frac{e^{-\lambda} \lambda^x}{x!} \\
= \sum_{u=0}^{\infty} (u + 1)^n \frac{e^{-\lambda} \lambda^{u+1}}{(u + 1)!} \quad \text{substituting } x = u + 1 \\
= \lambda \sum_{u=0}^{\infty} (u + 1)^{n-1} \frac{e^{-\lambda} \lambda^u}{u!} \\
= \lambda E[(X + 1)^{n-1}]
\]

\((5 \text{ marks})\)