Martingale Markets: Abstracting the Distinguished Asset

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Appealing Theory, Appalling Facts, and Eternal Hope

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- Part I: Rubinstein's Theorem and the CAPM

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- Part V: Return to the Menagerie and a Free Snack

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- Big CAPM Conclusion: For each asset A
- $$r_A - r_0 = \beta(r_M - r_0) + \epsilon \quad \text{where} \quad \epsilon \sim N(0, \sigma)$$

  where r_A is the return on the asset, r_0 is the risk free rate, r_M is the market return, and β is a constant that depends on all the utilities and on the distribution of the asset returns.

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- *Common Sense (of Sorts): One should only assume that which one cannot test and reject.*

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  - Modest Asymmetry — Left tail is fatter than the right tail

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- Second Stylized Fact: **Asset returns are not independent. At a minimum their squares show substantial predictability**

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- **The stochastic features of asset returns may possess many mysteries, but there are also consistent behaviors that are found across different nations, across different asset classes, and over many different time periods and time scales.**

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- There is much interesting history and sociology in the Black-Scholes trajectory.

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- **It is odd that we impute so much "efficiency" to markets where the biggest players are so tangled up in their own pajamas.**

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- **We can hunt for this model by leaning hard on generality and abstraction**

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- Given a constant $0 \leq k < \infty$, we say the triple $(\mathcal{T}, V, k)$ is a *martingale market* provided that for each $S = \{S_t\} \in \mathcal{T}$ the process $J_t(S, V, k)$ defined by

$$J_t(S, V, k) = S_t - k \int_0^t \frac{1}{V_u} \, d \langle S, V \rangle_u, \qquad 0 \leq t \leq T,$$

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- Admittedly, this is strange, but bear with me. I'll at least show it is interesting.

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- Three Nice Properties
  - V, the distinguished asset, is a submartingale.
  - $\log V$ is also submartingale
  - Uniqueness: If $(\mathcal{T}, V)$ and $(\mathcal{T}, V')$ are MarketGales, then there is a constant c such that $V_t = cV'_t$ with probability one for all $0 \le t \le T$.

First Interpretation of k

Consider the special case $dV_t = \mu_t V_t dt + V_t \sigma_t \cdot dB_t$ and calculate

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\begin{aligned}
dJ_t(V, V, k) &= dV_t - k \frac{1}{V_t} d \langle V, V \rangle_t \\
&= \left(\mu_t V_t - k V_t \sum_{i=1}^{d} \sigma_t^2(i) \right) dt + V_t \sigma_t \cdot dB_t.
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For J_t to be a martingale we need to have with probability one that

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- Instantaneous drift increases is linear in the risk aversion k
- The instantaneous drift increases as the the volatility increases.

Volatility Discounted Utility: Second View of k

For $1 < k < \infty$, the classical isoelastic utility function

$$U_k(w) \equiv \frac{w^{1-k}}{1-k} \tag{1}$$

has Arrow-Pratt relative risk aversion $-wU_k'(w)/U_k''(w) = k$.

Definition (Volatility Discounted Utility)

For a martingale market $(\mathcal{T}, V, k)$ and $1 < k < \infty$ the *volatility discounted utility*, $D(w) \equiv D_{k,V,t}(w)$, is the map from $\mathbb{R}^+$ to $\mathbb{R}$ that is defined by

$$D_{k,V,t}(w) \equiv U_k \left(w \exp(-\frac{k}{2}\langle \log V, \log V \rangle_t) \right).$$

The Martingale Properties of Volatility Discounted Utilities

Theorem (Discounted Martingale Theorem)

If the triple $(\mathcal{T}, V, k)$ is a martingale market, then for each $S \in \mathcal{T}$,

the process $\{D(S_t) : 0 \leq t \leq T\}$ *is a supermartingale and*

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- This is just what we want: The un-distinguished assets are not so good (in this particular sense)

The Martingale Properties of Volatility Discounted Utilities

Theorem (Discounted Martingale Theorem)

If the triple $(\mathcal{T}, V, k)$ is a martingale market, then for each $S \in \mathcal{T}$,

the process $\{D(S_t) : 0 \leq t \leq T\}$ *is a supermartingale and*

the process $\{D(V_t) : 0 \leq t \leq T\}$ *is a martingale.*

- This is just what we want: The un-distinguished assets are not so good (in this particular sense)
- The distinguished asset holds its own against the ravages of time and risk

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  - A Small inefficiency that yields a **Free Snack**

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- Criminal Theme: *Managed Distributions*

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- Less inclusive way: Extra Annual Fee of $4,300 for each million. Perhaps "not much" but why not keep it?

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  - Hence Your **Free Snack** — worth perhaps a $2K-$3K "bonus" year, just a nibble, but still, why not have a snack?