Waiting for Equilibrium: Derivative Pricing in an Unsteady World

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First: Some Ambidextrous Attitudes Toward Speculation

"While London's financial men toiled many weary hours in crowded offices, he played the market from his bed for half an hour each morning. This leisurely method of investing earned him several million pounds for his account and a tenfold increase in the market value of the endowment of his college, King's College, Cambridge." (B. Malkiel)

The Spirit of Speculation Has Been Part of a Long Tradition in Economics

- David Ricardo (1772-1823) made a fortune by speculating on bond prices before the battle of Waterloo.
- Irving Fisher (1867-1947) made a fortune by inventing the "rolodex." He then lost it all speculating in 1929.

A Two Part Plan with an Intermission

- Epistemology of Option Pricing
  - Three views of "Black-Scholes" as a social event, as applied mathematics, and as a science paradigm
- Refining the Models, Refining the Logic
  - Price models with observational errors
  - Arbitrage and predictability
  - Gibbs models and derivative pricing

The "Black-Scholes Formula" as a Social Event

- Development of the Options Markets
  - Post hoc ergo propter hoc?
  - Expansion of other "derivative" markets
- Development of Mathematical Finance
  - additional journals, expanded coverage of traditional journals
  - new programs, new curricular developments

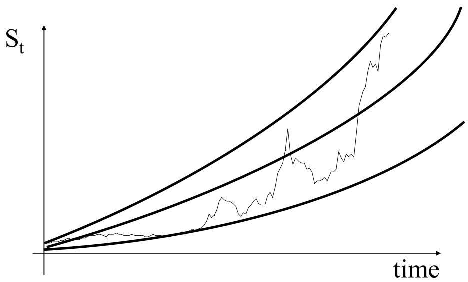
Pre-Step to the Applied Mathematics: The Black-Scholes (Samuelson?) Model

- Stock Model: $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$
- Bond Model: $d\beta_t = r\, \beta_t\, dt$

Note: S_t is a Markov process. It will eventually turn out that this is not an essential (or even an advisable) feature of a price process, but it seems hard to believe that the current theory could have evolved without starting here.

Geometric BM as a Price Model

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$



Pricing of European Call Option Under the Black-Scholes Model

- Time T pay-off: X=max $\{(S_T-K), 0\}$
- "Arbitrage Value": $\mathbf{S_t P_+ - K e^{-rt} P_-}$
- $P_{+/-}=\Phi(d_{+/-}/\text{sqrt}(\sigma(T-t))$
- $d_+= \log (S_t/K)+(r +\sigma^2/2)$
- $d_-= \log (S_t/K)+(r -\sigma^2/2)$

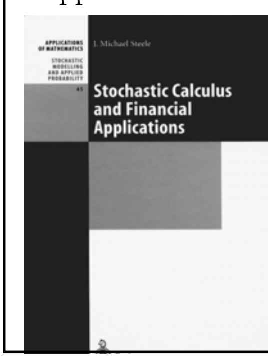
A Structural Observations: On Guessing the "Form" of an Option Formula

The Stock Model: $dS_t = \mu S_t dt + \sigma S_t dW_t$ and the Bond Model: $d\beta_t = r \beta_t dt$ make the couple (S_t ,β_t) a Markov Process.

The pay-off X=max $\{(S_T-K), 0\}$ just depends on the ending value S_T --- not the full path $\{S_t: 0<t<T\}$.

We can guess that the option price has a representation in the form $f(t,S_t)$.

"Black-Scholes" and the Epistemology of Applied Mathematics



Black and Scholes give two arguments for their pricing formula.

One of these is widely repeated and uses the Ito "analog" of (f/g)'=f'/g.

The other argument has "not been seen again."

The Famous Delta Hedge Argument

- In 1973 Black and Scholes follow a lead from *Beat the Market* by Thorp and Kassouf. Linearizing through the origin they consider the portfolio:

  $$X_t = S_t - f(t, S_t) / f_x(t, S_t)$$
- Ito's Formula with the odd $(\varphi/\psi)'= \varphi'/\psi$ twist
- Yields the Black-Scholes PDE
- Economic vs Mathematical Reasoning
- Motivation for a "PDE Model"

The Less Famous CAPM Argument

- Return on any asset will (in theory) be equal to the risk-free rate plus a multiple of the "return of the market" in excess of the risk-free rate.
- The multiplier is just the covariance of the asset return and the market return, divided by the variance of the market return (Beta).
- Apply this to S_t and $f(t, S_t)$ to get two equations. Clear the market, get the BS-PDE

Two Questions with (Partial) Answers

- What if you don't use $(\varphi/\psi)' = \varphi'/\psi$ in the delta hedge argument? What do you get?
- ***Answer***: You get a nonlinear PDE which must be in some sense approximated by the Black-Scholes PDE, but no one seems to have pursued this program.
- Why did the CAPM argument just disappear.
- ***Answer***: Because it was pure flim-flam. You can replace CAPM with a cubic or quadratic and the "argument" goes through.

Where the Arguments Took Us First

- Empirical performance is not particularly good.
- The Delta hedge idea has serious impact on the practical world of finance.
- The two motivational arguments of Black and Scholes have been supplemented by more satisfying arguments by Merton, and especially by Harrison and Kreps.
- The Harrision-Kreps Martingale theory now almost completely eclipses the PDE theory.

Where the Arguments May Be Taking Us Now

- More people are trying to face up to the defects of the underlying stock price model.
- Martingale theory has led to a more directly empirical approach to derivative pricing: the Theory of Pricing Kernels. It avoids modeling of the underlying security prices.
- The basic logic of derivative pricing theory is being revisited.

An Observation Model for Prices: The Qualitative Case

- "Price" seems so obvious, one may not think to ask if it is really well defined. Yet is it really?
- Still, just for the sake of argument, if you consider the possibility of "observational error" in prices, you are led to many natural generalizations of the Black Scholes Model.
- Some of these may be useful, or at least instructive.

A Observational Model for Stock Price: One Concrete Possibility

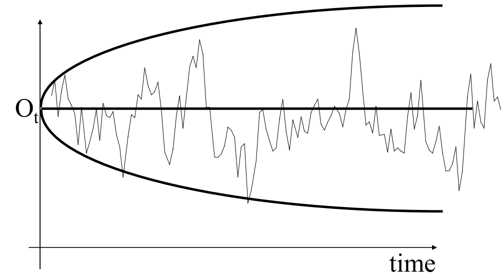
- BM with Drift: $dX_t = \mu\, dt + \sigma\, dW_t$
- A Model for Wobble: $dO_t = -\alpha\ O_t\, dt + \varepsilon\, dW_t'$
- A Model for Price: $S_t = S_0 \exp(X_t + O_t)$

The point is that S_t is essentially geometric Brownian motion, but with a mean reverting observational error.

Please Note: S_t is NOT a Markov process

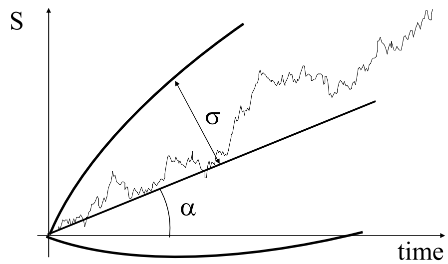
Mean Reverting Process

$$dO_t = -\alpha O_t\, dt + \varepsilon dW_t'$$



Ordinary BM with Drift

$$dX_t = \mu\, dt + \sigma\, dW_t$$



What Do We Do? What Do We Get ?

- The PDE approach is meaningless in this context, but …
- Martingale Pricing Theory is up to the task. An easy exercise gives one a formula for the price of a European call option.
- It may be a bit surprising that you get EXACTLY the Black-Scholes formula,
- *Except* that the old σ is replaced by a function of the new model parameters.

What Does this Tell Us (Part I)

- BM with Drift: $dX_t = \mu\, dt + \sigma\, dW_t$
- A Model for Wobble: $dO_t = -\alpha\ O_t\, dt + \varepsilon\, dW_t'$
- A Model for Price: $S_t = S_0 \exp(X_t + O_t)$

Even though S_t is not a Markov process, the pricing formula is of the form $f(t, S_t)$.

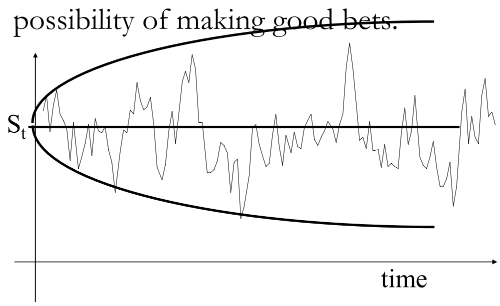
This is amusing … but perhaps not stunning.

What Does this Tell Us (Part II)

- BM with Drift: $dX_t = \mu\, dt + \sigma\, dW_t$
- A Model for Wobble: $dO_t = -\alpha\ O_t\, dt + \varepsilon\, dW_t'$
- A Model for Price: $S_t = S_0 \exp(X_t + O_t)$

What is more stunning is that for large ε and small σ our process is marvelously predictable. Draw the picture in your mind.

For small σ and big ε you get …a mean reversion picture, which suggests the possibility of making good bets.



Why Do We Believe What We Believe?

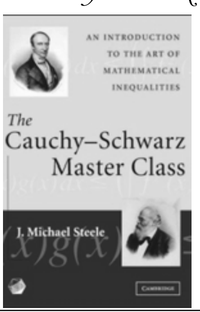
- This question can get under your skin if you let it.
- Here we have a price process which contains both the BS model and one that is highly predictable.
- Martingale pricing theory applies seamlessly as we move from one extreme to the other.
- Can we have appropriately priced options under a model which makes every man a king?

Some Final Considerations (But, *Maybe*, one Slide to Come…)

- The ISI Citation Index gives just under 2000 citations of BS (1973).
- The mathematical theory of option pricing has achieved a certain maturity…
- but more than a few mysteries remain.
- Observational models are a huge field in their own right with a vast technology.
- Price modeling is very active and many empirical issues are still poorly understood.
- Thanks!

MSO: Can You Find the Day When Martha Was Convicted?

A Trailer: *The Cauchy-Schwarz Master Class*



- "A three-hundred page book about a one-line inequality"
- Coaching for problem solving, plus all of the classical inequalities viewed with new eyes
- The real truth about Bunyakovsky…