A Drunk, Her Dog and A Boyfriend: An Illustration of Multiple Cointegration and Error Correction

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If the linear combination of non-stationary random variables results in a stationary series then the combined variables can be described as cointegrated. This article extends the humorous example introduced by Murray (1994) which related the description of a cointegrated pair, a drunk and her impetuous dog, who adjust their paths so as to avoid straying too far apart. We consider the meandering of a drunk, her dog and a boyfriend as an illustration of multiple cointegration and error correction. This generalisation beyond the bivariate case opens up rich array of possibilities but introduces complications to the identification and estimation of the system.

KEY WORDS: Drunkard's Walk, Nonstationary Process; Econometrics; Cointegrating Vector; Johansen.

1. INTRODUCTION

The example of a drunk and her dog introduced by Murray (1994), henceforth denoted MM, has been proving to be a valuable expository tool for demonstrating of the behaviour of a cointegration in bivariate time series. It links tidily with the traditional example of the meandering or 'random walk' of a drunken man used to demonstrate a simple non-stationary process: it portrays a readily visualisable example of cointegration and its corequisite feature of error correction. Here we extend the illustration by introducing a third participant to the party; the inebriated boyfriend. The introduction of this third participant raises the possibility of more than one cointegrating combination and a more complex system of error corrections than can arise in the pairwise case. The unravelling of the cointegrating relationships among several

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variables and the estimation of their associated error correction processes have been the subject of much recent research in the econometrics (Muscatelli and Hurn 1992).

2. THE TALE OF A DRUNK, HER DOG AND A BOYFRIEND

In a scenario similar to that used by MM, we begin with the drunk who wanders around tracing a random walk path:

$$x_t - x_{t-1} = u_t \tag{1}$$

while her dog responds to the sound of her voice and adjusts its meandering according to how distant she sounds:

$$y_t - y_{t-1} = -\alpha(y_{t-1} - x_{t-1}) + v_t \tag{2}$$

Here her dog, named Oliver, adjusts the distance between his current position and his previous position $(y_t - y_{t-1})$ in proportion α to his distance $(y_{t-1} - x_{t-1})$ from his mistress. The expression $(y_{t-1} - x_{t-1})$ captures the cointegrating, or long run, relationship of Oliver with his mistress. This cointegrating relationship is an expression of his desire to be in the same place as the drunk; i.e. he desires $(y_t = x_t)$. This shows the essential "attractor" feature of cointegration; that at least one of the participants is attracted to the other, such that their distance apart $(y_t - x_t)$ traces a stationary series.. In equation (2) the dog is responding to his mistress' voice by changing his position to move towards her. He assesses his distance from his mistress in period t-1, measured as $(y_{t-1} - x_{t-1})$, and plans his next step, $(y_t - y_{t-1})$, so as to reduce this distance, or error. We have measured the distance as $(y_{t-1} - x_{t-1})$ rather than $(x_{t-1} - y_{t-1})$, therefore Oliver's attraction to his mistress will yield a negative response as his movements tends to reduce the distance. Thus equation (2) is termed an error correction mechanism (ECM). Granger’s Representation Theorm (Granger 1983) has shown that whenever cointegration exists between non-stationary series there must exist an ECM maintaining it.

The speed of adjustment parameter α will take a value between one and zero. If Oliver is inclined to roam considerably with only a weak desire to return close to her, then the value of α will be close to zero. If, however, Oliver is an attentive dog with a strong desire to stay close, then α will be closer to unity. Example of these two scenarios, generated by Monte Carl simulation are depicted in Figures 1 and 2. Each figure displays a contrast of the distance between Oliver and his mistress as they meander from the bar. They can be interpreted as showing their progress over say the first 400 seconds (or steps at one step per second).

In Figure 1; an attentive Oliver ($\alpha = 0.3$) implies that their distance apart is never very far and their paths cross (i.e. ($y_t - x_t$) changes sign) frequently. In Figure 2, showing an inattentive Oliver ($\alpha = 0.05$), their distance apart can grow large at times although convergence always occurs eventually. The number of times their paths cross is much less frequent than depicted in Figure 1. Thus, we may interpret the α parameter as the 'strength of attraction' of

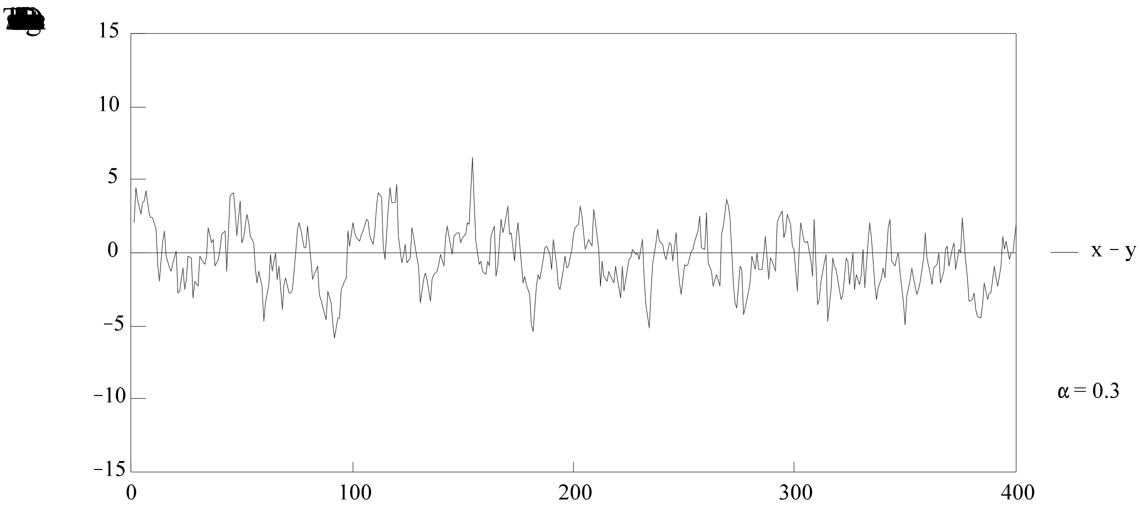


Figure 1 Distance Apart of the Drunk and Her Dog when α = 0.3

variable while the progress of her dog is dependent on her position: there is therefore a causal relationship (in the Granger [1999] sense) from the drunk to her dog., but there is no causation in the reverse direction

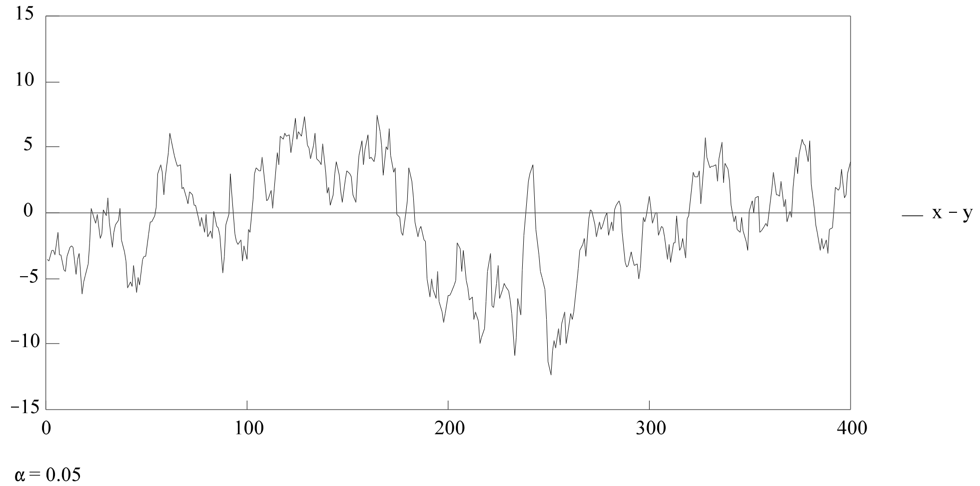


Figure 2 Distance Apart of the Drunk from Her Dog

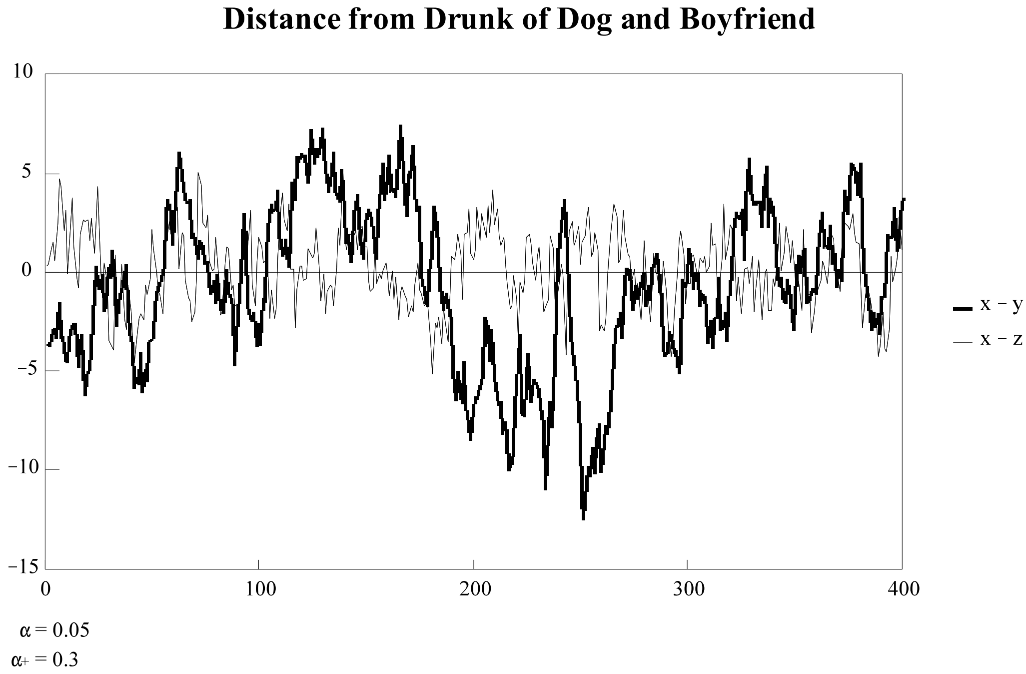
We now introduce into our scenario a boyfriend, Kinley, who we will initially consider to be attracted to the drunk but indifferent to Oliver. Kinley desires to be close to his girlfriend but in the nature of this stochastic process does not hold her hand, just as Oliver is not held on a leash. Kinley's drunken progress can be represented by:

$$z_t - z_{t-1} = -\alpha^{+}(z_{t-1} - x_{t-1}) + w_t \qquad \textbf{(3)}$$

Here we see that Kinley is adjusting his step (z_t - z_{t-1}) to move towards his girlfriend; his speed of adjustment is given by the value of α^{+}. A value close to one would indicate a strong desire by

Kinley to stay close to her, while a value close to zero would suggest that Kinley was readily distracted! The meanderings of our happy trio are displayed in Figure 3 where Kinley behave as an attentive boyfriend with $\alpha^+ = 0.3$, while Oliver roams widely with $\alpha = 0.05$.

Figure 3



Our scenario has been kept deliberately simple. The meandering of the mistress determining the paths of both her dog and her boyfriend, who are totally indifferent towards each other. She is not interested in either Kinley or Oliver but attractiveness to them ensures that the cointegrated triple stay close together. Note that the time paths of the three will each appear nonstationary, when viewed separately.

It is important to understand that this particular exposition places some restrictions on the model which will often be inappropriate in economic applications. This is because the interpretation of our trio as a cointegrated triple requires expression of the equations in terms of distances. For example, equation (3) gives the distance between Kinley's positions at two points in time (or between two steps) as a function of of the distance between him and the drunk. This imposes binary coefficient values within the cointegrating relationship. It is constrained to (y_{t-1} - x_{t-1}) rather than the more general ($\beta_1 y_{t-1}$ -β_2 x_{t-1}) where β_1 and β_2 are some unrestricted coefficient values. Secondly, the distance measure constrins the cointegration relationship to be of a bivariate nature, e.g. a cointegrating relationship of the form $(y_t - x_t - z_t)$ is not feasible.

However, we can extend our illustration to include some come complex interactions. The drunk may reciprocate Kinley's affection for her and adjust her progress so that equation (1) becomes:

$$x_t - x_{t-1} = -\alpha^{**} (x_{t-1} - z_{t-1}) + u_t \qquad \textbf{(4)}$$

If the drunk adjusts her walk to respond to the positions of both Oliver and Kinley then equation (4) must be extended to allow for both sources of attraction:

$$x_t - x_{t-1} = -\alpha^{*} (x_{t-1} - y_{t-1}) - \alpha^{**} (x_{t-1} - z_{t-1}) + u_t \qquad \textbf{(5)}$$

Here the drunk’s behaviour is affected by two independent cointegrating relationships; she responds to her distance from the dog and to her distance from her boyfriend. The values of α^* and α^{**} revealing the strengths of her adjustments to Oliver and Kinley respectively, so if $\alpha^* > \alpha^{**}$ she exhibits a stronger attraction to the pooch than the boyfriend!

One can readily extend the interactions further to allow, say, Kinley to have an attraction, or even an aversion, to Oliver:

$$z_t - z_{t-1} = -\alpha^{+} (z_{t-1} - x_{t-1}) - \alpha^{++} (z_{t-1} - y_{t-1}) + w_t \qquad \textbf{(6)}$$

If Kinley dislikes the dog, then the 'strength of attraction' to it, α^{++}, would be negative. Figure 4 shows the outcome of such a case with the boyfriend maintaining close proximity to the drunk while his aversion to the dog is revealed by his tendency to be on the opposite side, so the drunk is usually between them.

Having an error correction coefficient outside the zero to one range creates the possibility of the series diverging, rather than remaining close together. In such a case, cointegration cannot be maintained unless the magnitude of the attractors is sufficient to outweigh the repulsion.

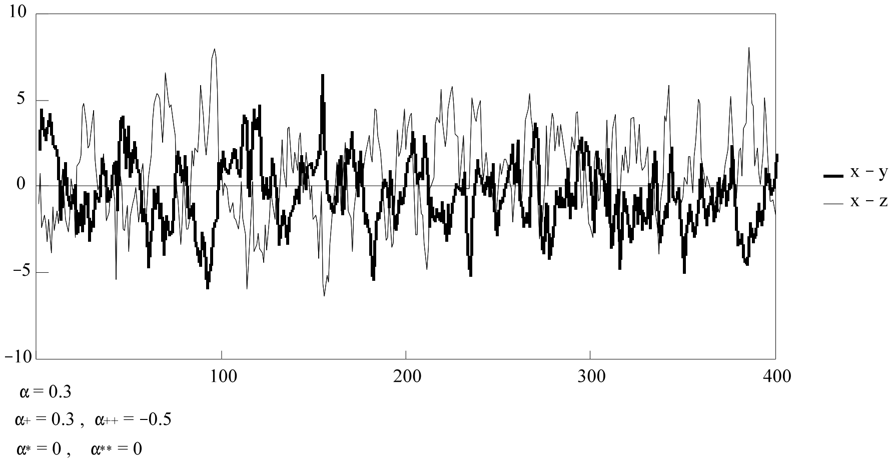


Figure 4 Distances between the drunk and her dog and her boyfriend when $\alpha = 0.3$, $\alpha^+ = 0.3$, $\alpha^{++} = -0.5$, $\alpha^* = 0$ and $\alpha^{**} = 0$

In (6), we introduce a third cointegrating relationship (z_t - y_t), so that Kinley's position is now affected byhis (lack of) affection for Oliver. However, the relationship is just a linear combination of the other two, (z_t - x_t) and (x_t - y_t). Thus, although our system appears to containthree cointegrating relationships, there are only two which are linearly independent. For example , we can eliminate (z_t - y_t) as a cointegrating relationship by rewriting (6) as

$$z_t - z_{t-1} = -(\alpha^+ + \alpha^{++})(z_{t-1} - x_{t-1}) - \alpha^{++}(x_{t-1} - y_{t-1}) + w_t \qquad \textbf{(7)}$$

With three variables it is not possible to have more than two cointegrating relationships.

3. ESTIMATION

So far we have considered the reaction processes and the time paths they would generate but what if you only witnessed the outcome of these cointegrated processes? To the casual observer of the girl, the dog and the boyfriend meandering along, their tendency to stay together would be apparent after a time. However, it may not be so obvious who, if any, was taking the lead, who was responding and in what way. The observer may wish to glean some insights into the relationships between the trio.

Starting with the simple bivariate system, such as used by MM, the estimation of the cointegration relationship and the accompanying error correction mechanism can be achieved with simple regression. For data generated from a system comprising equations (1) and (2) displayed in figure 1 the cointegrating regression yields:

$$y_t = \underset{(0.00)}{0.97x_t} + \underset{(0.10)}{0.051} \; e_t \qquad \textbf{(8)}$$

$$R^2 = 0.98 \quad DF = -8.473 \quad CRDW = 0.62 \quad T = 400$$

where conventional standard errors are in brackets and T is the sample size. The Dickey-Fuller (DF) and the Durbin-Watson (CRDW) statistics for testing cointegration (Engle and Granger 1987) are also given.

The true cointegrating parameter is 1.0 and so our regression estimate appears to be very accurate. However, the consistency of the ordinary least squares estimatorhere is dependent on the existence of a cointegrating relationship, i.e. on the errors being stationary (Banergee, et.al. 1993). Note that conventional t-tests on the regression coefficients cannot be used to establish cointegration, as the conventional standard errors from regressions involving nonstationary data are grossly underestimated and inferences based on them will likely lead to erroneous conclusions. This can give rise to a “spurious regression”, where there appears to be a significant relationship between the variables even though there is not(Granger and Newbold 1974, Yule 1926). A spurious regression is characterised by its non-stationary errors. The Df and CRDW statistics can be used to test the null hypothesis that the disturbances are nonstaionary against a stationary alternative. Using 10% criticl valuesof -3.02 for the DF test and 0.16 for the CRDW test implies that the null is rejected by both tests in this example. The critical values were obtained from Engle and Yoo (1987). In more a complex system, it may be necessary to use an Augmented Dickey-Fuller (ADF) statisitic to conduct this test (Banergee, et.al. 1993). The augmentation is designed to filter out any serial correlation in the errors of the test equations.

The regression estimation of the error correction mechanisms involves stationary series and yields equations (9) and (10):

$$\begin{aligned} \Delta y_t &= -\ 0.3015\, e_{t-1} - 0.070 + v*_t \\ &\qquad (0.026) \qquad (0.048) \\ R^2 &= 0.26 \qquad DW = 2.00 \end{aligned} \tag{9}$$

$$\begin{aligned} \Delta x_t &= \ 0.008\, e_{t-1} - 0.066 \ ++ \ v_t^* \\ &\qquad (0.026) \qquad (0.05) \\ R^2 &= 0.0002 \qquad DW = 1.992.0 \end{aligned} \tag{10}$$

where e_t is the residual, or error correction term, from the cointegration regression of equation (8) above. These regressions provide very accurate estimates of the original error correction parameters; revealing Oliver's attraction to his mistress in equation (9) and his mistress's indifference to Oliver with the negligible and statistically insignificant regression of equation (10). Thus, the observer has been able to establish the long run cointegrating relationship and to determine the dynamic adjustment, showing that in this case it is Oliver responding to his mistress but she is oblivious to him.

Turning now to a system involving all three participants depicted in equations (1), (2) and (3). The observer may wish to glean some insights into the relationships between the three: "Is Kinley merely following the dog and not really interested in the girl?", "Are these two independent dog lovers following the pooch?" or, "Has Oliver taken a liking to a drunken Kinley while Oliver's mistress is following merely to ensure that she does not lose her beloved dog?". These and other exciting possibilities would entertain the minds of any statistically minded Sherlock Holmes (Doyle 1909).

Unravelling the potentially complex interactions between the three participants is analogous to the detective work required in unravelling econometric relationships between non-stationary economic time series. The series appear to evolve together through time but one would like to know the values of the parameters of both these long run cointegrating relationships and the ECM. Unfortunately, estimation of the multivariate system poses considerably more complications then the bivariate example previously considered. This is mainly caused by the possibility that there could be zero, one or two cointegrating relationships, rather than just zero or one as in the bivariate case.

4. THE JOHANSEN MAXIMUM LIKELIHOOD TECHNIQUE

Mathematically, any linear combination of the cointegrating relationships is equally valid, but it is never possible to write more than two of them such that they are all linearly independent. Thus we have up to two general cointegrating relationships of the form:

$$\begin{aligned} (\beta_{11}x_t + \beta_{12}y_t + \beta_{13}z_t) &= \epsilon_{1t} \\ (\beta_{21}x_t + \beta_{22}y_t + \beta_{23}z_t) &= \epsilon_{2t} \end{aligned} \qquad \textbf{(11)}$$

Unfortunately, OLS is only validwhen the system contains but one cointegrating relationship. If two exist, it poses a tytpe of identification problem akin to that encountered in the conventional problem of estimating simultaneous equations. In practice, one is unlikely to have a strong prior about the number of cointegrating vectors and therefore must estimate this via specific hypothesis tests.

Our system of up to two cointegrating relationships can be written in matrix form as:

$$\beta' X_t = E_t \qquad \textbf{(12)}$$

where β is the (2×3) matrix of cointegrating parameters, X_t is the vector of three cointegrating variables and E_t is the vector of cointegrating errors usually referred to as the error correction terms.

The ECM associated with this system is:

$$\Delta X_t = \alpha E_{t-1} + U_t \tag{13}$$

where α is a (3×2) matrix of error correction parameters and U_1, U_2,.... U_T are independent, normally distributed error terms. Alternatively, the complete error correction mechanism together with its system of cointegrating equations could be written as:

$$\begin{aligned} \Delta X_t &= \alpha\beta' X_{t-1} + U_t \\ &= \Pi X_{t-1} + U_t \end{aligned} \tag{14}$$

Complications arise because the model in (14) is over parameterised. Although Π can be estimated, it is not possible to solve uniquely for its components, α and β. However, the Johansen maximum Likelihood (ML) method can be utilised to gain insights into the properties of α and β (Johansen 1991, Johansen and Juselius 1990). The first task is to determine whether the number of cointegrating relationships, p, is zero, one or two. Given that, in this context , the individual data series are nonstaionary, it must be the case that p<3 and therefore that α and β have less columns than $\Pi = \alpha\,\beta`$. Thus Π is the product of two smaller matrices and has reduced rank p. An intuitively reasonable approach here would be to calculate $\hat{\Pi}$, the unrestricted OLS estimate of Π, and determine the number of its eigenvalues that are significantly different from zero. The number of non-zero eigenvalues would then be equal to p, the cointegrating rank. Alternatively, one could calculate the singular values of $\hat{\Pi}$ (i.e. the square root of the eigenvalues of $\hat{\Pi}'\hat{\Pi}$) and test their significance from zero, since they are guaranteed to be real and nonnegative. Johansen's trace test is based on the singular values of a normalised version of $\hat{\Pi}$. This normalisation is derived via the Maximum Likelihood technique and it ensures that the test statistics have asymptotic distributions that are free of nuisance parameters. The trace test is identical to a conventional likelihood ratio test.

To illustrate how the JML procedure unravels the underlying structure behind multivariate time series, consider three data series generated by the system comprising equations 1, 2 and 3. It yields trace test statistics given in Table 1 showing that the null hypotheses of 'rank zero' and 'rank one or less' can be rejected in favour of 'rank equals two'.

Table 1 Johansen's trace statistics

H_0: rank=p	Trace Statistic	90% critical value
p = 0	154.2*	28.4
p <= 1	19.59*	15.6
p <= 2	0.03467	6.69

Note: * significant at the 10% level

Corresponding to each of the two significant eigenvalues is an eigenvector whichcontain the Maximum Likelihood estimates of the cointegrating vectors β_j (Johansen 1991, Johansen and Juselius 1990). These eigenvalues are given in Table 2 where they have been arbitrarily normalised on their diagonal elements. The estimates were obtained using the PC-FIML econometrics package (Doornik and Hendry 1994) and imposing the restriction that p=2.

Table 2: Standardised Eigenvalues

	x_i	y_i	z_i
First eigenvector β_1	1.00	0	-0.99
Second eigenvector β_2	0	1.00	-0.90

When multiplied by X_t each eigenvector yields one linear combination of the variables which is stationary. However, any linear transformation of these eigenvectors will also yield stationarity. In the context of the drunk (x_i) , her dog (y_i) and boyfriend (z_i) we know that the drunk and the boyfriend are cointegrated and that the true values of the cointegration vector are (1, 0, -1). Thus, given the noisy[2] nature of the system, it seems that the estimates of (1, 0, -0.99) for this first cointegrating vector are reasonably accurate. The true values for the second eigenvector are (1, -1, 0) and it may seem that our estimated values (0, 1, -0.90) are grossly inaccurate. However, this apparent impression arises from the arbitrary choice to normalise on y_t. Identification of β is only possible down to a linear combination of the cointegration vectors, so any linear transformation of β_1 and β_2 is equally acceptable as a cointegrating vector. By subtracting β_2 from β_1 we can create a new cointegrating vector, $\beta_2^{`}$ = (1, 1, -0.99), which although equally valid mathematically, now conforms to the "true" normalisation, representing the cointegration between the drunk and the dog. The practising econometrician faces similare problems and must rely on economic theory when choosing appropriate normalisation of his/her cointegrating vectors.

Table 3 contains the JML estimates of the matrix α in equation (14) , using the data shown in Figure 3 and the estimated cointegrating vectors β_1 and $\beta_2^{`}$ . Recall that $\alpha$ measures the strength of attraction, i.e. it contains the error coefficients, It turns out that $\hat{\alpha}\hat{\beta}^{`}$, where $\hat{\alpha}$ and $\hat{\beta}^{`}$ are the JML estimates of α and β respectively, is the closest rank p matrix to the full rank least squares estimate $\hat{\Pi}$.

[2] Readers should note that the term "noisy" does not refer to the barking of the dog, but to the relative strength of the random error component.

Table 3 Estimated Error Correction Coefficients

Dependent Variable	Coefficients on First Eigenvector		Coefficients on Second Eigenvector	
	Estimates	True Values	Estimates	True Values
x_i	0.027	0	-0.016	0
y_i	-0.044	0	0.063	0.05
z_i	0.34	0.3	-0.035	0

Alternative methods of estimating cointegration vectores as part of the ECM have been advocated by Hendry (Banergee et al 1993, Gilbert 1986). The Hendry method estimates a general dynamic specification and reduces it, through sequential testing, to a parsimonious specific form with cointegrating vectors contained within the ECM. The choice of the appropriate methodology for estimating cointegrating vectors within dynamic models remains contentious (Inder 1993; Muscatelli and Hurn1992).

5. CONCLUSION

An extension of the original illustration of the drunk and her dog to include a third participant, the boyfriend, could be further expanded to include other non-stationary participants, the boyfriend's dog for instance. Such an expansion while perfectly feasible does not open up further conceptual insights beyond those raised by the three variable case and, consequently, has not been pursued here.

While the illustration of the drunk and her dog can yield some valuable insights into the nature of non-stationary cointegrating processes it does carry with it some limitations. Specifically, the distance measures impose binary parameter value and bivariate cointegrating vectors, whereas in economic systems the cointegrating vectors would not be so constrained.

Further extensions of the illustration could be pursued to investigate the effects of aggregation: with temporal aggregation where the data may have been observed at intervals longer than one second, say ten second intervals, no serious complications would arise for either the underlying process or the estimation of the cointegrating vector(s) and error correction mechanism(s). The estimate os the cointegrating vectors will be consistent although the estimates of the ECM will reflect the ten period aggregation, with the speed of adjustment coefficient measuring the adjustment towards the long run equilibrium over ten second not one. However, temporal aggregation does introduce complications where more frequent observations can lead to seasonal and periodic features in the cointegrating vectors and the ECM (Franses 1994; Hylleberg, Engle, Granger and Yoo 1990).

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