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# Notes, Comments, and Letters to the Editor

## The St. Petersburg Paradox: A Con Game?*

Careful analysis of the "St. Petersburg" lottery reveals no logical or mathematical absurdity inherent in risk neutrality for money. There is an empirical absurdity, but it rests on an additional, easily overlooked assumption about the gullibility of the gambler that is itself empirically absurd.

Since the time of the Bernoullis,[1] an impression has lingered that the famous "St. Petersburg game" is some kind of counterexample to risk neutrality for money, and that some kind of mathematical or logical contradiction inheres in the use of expected monetary values as a utility index.[2] If this were, in fact, the case, their use could be not only inaccurate in practice but unsound in principle, and perhaps unsuited even for simplified decision models where small quantitative inaccuracies would ordinarily be tolerated. It is hoped that the following brief discussion of the underlying logic of the "St. Petersburg paradox," reinforced by a few mundane numerical illustrations, will help to lay these doubts to rest.

Let us "walk through" the paradox, step by step:

(I) First, one supposes that a "rational" person has been found whose utility for money happens to be both linear and risk neutral.

(II) One then proposes, in return for his payment of an entrance fee, to toss coins with the subject until he fails to win, and then to pay him $2^k$ cents, where $k$ ($= 0, 1, 2,\ldots$) is his total number of wins.

(III) One then calculates, from (I) and (II), that his utility for this proposition must be infinite:

$$\tfrac{1}{2} \cdot 1 + \tfrac{1}{4} \cdot 2 + \tfrac{1}{8} \cdot 4 + \tfrac{1}{16} \cdot 8 + \cdots = \infty,$$

and in particular that it exceeds \$1,000 (or any other finite amount).

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[1] See especially Daniel Bernoulli's 1738 paper [1]. The paradoxical game (or gambling proposition) was propounded 25 years earlier by Nicolas Bernoulli in a letter to Pierre de Montmort; this letter is reproduced in [1], along with a subsequent analysis of the problem by Gabriel Cramer.

[2] See, for example, the opening sentence of [5].





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The paradox is not yet upon us. An infinite utility may not be absurd, per se, when unboundedly large sums of money are in prospect.[3] Let us move ahead, then, cautiously:

(IV) One now argues, from (III), that the subject will gladly pay \$1,000 (or any other amount) as an entry fee.

(V) One next observes, from experience or introspection, that rational people most certainly do *not* behave as in (IV).

(VI) Finally, one concludes from (I) and (V), that linear risk-neutral utility for money is contrary to experience.

This, then, is the paradox. The reductio ad absurdum is not logical or mathematical, after all, but empirical. Our experience is contradicted. The conclusion is nevertheless impressive. At first glance, it throws serious doubt upon the realism of our starting assumption (I), namely, that rationality and risk linearity for money are compatible human traits.

A closer inspection, however, reveals a weak link in the chain—a fatal flaw that invalidates the conclusion. Despite our show of caution in moving from (I), (II), (III) to (IV), we left implicit one key assumption, without which we cannot claim to enter the mind of the subject and say what he will do, gladly or otherwise. The missing link:

(II-$\frac{1}{2}$) One assumes that the subject believes the offer to be genuine, i.e., believes *that he will actually be paid, no matter how much he may win.*

Something like this is essential. Since the end of the chain is empirical, we must provide (II-$\frac{1}{2}$) to link the fictive experiment to its real context. If it should prove, by the very nature of (II), that no rational person could possibly be convinced that the game is in earnest, then (II-$\frac{1}{2}$) fails, and with it (IV) and the conclusion (VI)[4].

[3] If the only issue were infinite utility, then we could obtain the desired effect in a probability-free context. Indeed, consider the game of "Blank Check," in which a sponsor offers, in return for a specific entrance fee, to pay the subject any finite amount of money that he names.

[4] See [3, p. 228]. Lord Keynes, in his well-known essay [4], also shows some concern for the believability of the game: "We are unwilling to be Paul, partly because *we do not believe Peter will pay us if we have good fortune in the tossing*, partly because we do not know what we should do with so much money ⋯ if we won it, partly because we do not believe we ever should win it, and partly because we do not think it would be a rational act to risk an infinite sum or even a very large finite sum for an infinitely larger one, whose attainment is infinitely unlikely" [4, p. 1370] (italics ours). But he apparently does not consider the disbelief in payment to be sufficient in itself to escape the paradox: "... Peter has undertaken engagements which he cannot fulfill; if the appearance of heads is deferred even to the 100th toss, he will owe a mass of silver greater in bulk than the sun. But this is no answer. Peter has promised much and a belief in his solvency will strain our imaginations; but it is imaginable" [4, p. 1368]. Here he seems to be saying that it is difficult, but possible, to

This may seem like nit picking, but it is not. The sting of the infinite series is only in its tail, and our experiment must be able to distinguish the *unbounded* proposal (II) from any bounded replica of it. Imagine for example, that (II-$\frac{1}{2}$) falls short—but just barely. Imagine that we have found a suitably rational, suitably risk-linear, and suitably mathematically educated personage, who lacks only the childlike trustfulness of (II-$\frac{1}{2}$). He believes, let us say, that the well-dressed stranger with pointed shoes is good for the value of the United States gross national product, say $2^{47}$ cents, but no more! Then the expected payoff, exclusive of entry fee, works out as follows:

$$\tfrac{1}{2}\cdot 1+\tfrac{1}{4}\cdot 2+\tfrac{1}{8}\cdot 4+\cdots+\frac{1}{2^{47}}\cdot 2^{46}+\frac{1}{2^{47}}\cdot 2^{47}=47\cdot\tfrac{1}{2}+1=24.5\text{ cents.}$$

So our paragon of linearity and gullibility would *not* gladly pay \$1,000; he would even balk at paying 25 cents for a chance at a year's national output!

Coming down to earth, let us suppose that a gambler is convinced that the casino will pay up to a \$10,000 limit on the St. Petersburg game. Then in the penny version (II), the "fair" entrance fee is just about 11 cents. In a dollar game, it would be \$7.61. These numbers are *not* absurd on their face. In short, the conclusion (IV) cannot be sustained when confronted by reasonable inputs and beliefs, and the paradox evaporates.[5]

Another way to drive home our point is to focus on the notion of "rationality" that enters into (I) and (V) above. There seems little doubt that anyone who would gladly pay \$1,000 to enter the St. Petersburg lottery is a fool. His folly is revealed, however, not in his linear attitude toward monetary gambles for high stakes—an attitude which may be foolish but which is not seriously tested on this occassion—but rather in his blissful confidence that the sponsors of the game can and actually will pay the prize, even to the tune

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imagine a "Paul" for whom the paradox works, in the sense of (II-1/2). We would rather exclude such freakish personalities from our models.

Karl Menger, in his otherwise lucid and well-argued treatment [6], is equally unsatisfactory on this point, sidestepping the credibility question with: "If he is sane, [the gambler will not] risk all or even a considerable portion of his wealth in a St. Petersburg Game" [6, p. 212]. This is similar to Bernoulli's: "any fairly reasonable man would sell his chance with great pleasure, for twenty ducats" [1, p. 31]. Neither author pauses to ask whether a sane man would believe in payoffs without bound.

[5] Menger rather stiffly regards the assumption of a payoff limit as violating the "definition of the game" and as "introducing factors that are outside the problem" [6, p. 214]. This is a curious objection from one who has just taken such care to stress the empirical content of the paradox. Moreover, he seems to regard bounding the payoff as no more than a device to ensure finiteness, saying that is is "unable to explain the remarkable discrepancy between mathematical expectation and actual behavior" (p. 214). Here he quite misses the point, made so well by Gabriel Cramer two centuries before, namely, that imposing even an outlandishly high bound will make the mathematical expectation acceptably small. (See [1, pp. 33–35].)

of $10^{60,204}$ dollars.[6] Compare him with the classic "sucker" who buys the Brooklyn Bridge for $272 (or whatever else he happens to have in his pocket at the time); the latter is also a fool, but not because the Brooklyn Bridge is not worth $272. Is the St. Petersburg paradox then nothing more than a "con" game?

One can, of course, devise other lotteries based on infinite series that diverge even more slowly.[7] They would only strengthen our present thesis, to wit: (1) there is no logical or mathematical absurdity in risk neutrality for money or income, and (2) the empirical "absurdity" that seems to arise depends on an additional, commonly overlooked assumption about the subject's credulity—an assumption that is itself empirically absurd. We conclude that the St. Petersburg game is a flimsy weapon indeed with which to attack the use of expected monetary values in decision theory, game theory, or economics, however vulnerable that usage may be on other counts.[8]

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[6] This is the "house limit" that yields a $1,000 expected value in the penny game.

[7] This is done, e.g., by Menger to show that any unbounded utility function—including logarithmic or "Bernoullian" utility—can be confronted by a lottery of the St. Petersburg type having infinite expected utility. (See [6, pp. 217–218]; incidentally, on p. 217, $e^{2n}$ should be $e^{2^n}$.)

[8] Essentially the same argument was previously made by Fry [2]; we are indebted to R. W. Hamming for this reference.