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Samuelson's Fallacy of Large Numbers and Optional Stopping

Erol A. Peköz

Abstract

Accepting a sequence of independent positive mean bets that are individually unacceptable is what Samuelson called a fallacy of large numbers. Recently, utility functions were characterized where this occurs rationally, and examples were given of utility functions where any finite number of good bets should never be accepted.[1] Here the author shows how things change if you are allowed the option to quit early: Subject to some mild conditions, you should essentially always accept a sufficiently long finite sequence of good bets. Interestingly, the strategy of quitting when you get ahead does not perform well, but quitting when you get behind does. This sheds some light on more possible behavioral reasons for Samuelson's fallacy, as well as strategies for handling a series of sequentially observed good investments.

Introduction and Overview

Samuelson (1963) told a story in which he offered a colleague a better than 50-50 chance of winning \$200 or losing \$100. The colleague rejected the bet, but said he would be willing to accept a string of 100 such bets. Samuelson argued that the colleague was irrationally applying the law of averages to a sum, and this perhaps has led to a more widely held perception that accepting a sequence of good bets when a single one would be rejected is a "fallacy of large numbers."

Since then a number of authors have studied this phenomenon. Samuelson (1989) gave examples of utility functions where a single bet is unacceptable but a sufficiently long finite sequence of good bets will be accepted. Also given were utility functions where a long sequence of good bets is never acceptable: Consider the utility function $U(x) = -2^{-x}$ and bets giving a 50 percent chance of losing \$1 or winning $\$(1 + \epsilon)$, for a sufficiently small $\epsilon > 0$. It can be shown that expected utility decreases with each additional bet made, even though the bets are favorable and the utility function is increasing. Pratt and Zeckhauser (1987) studied the related property they labeled "proper risk aversion," where investors unwilling to make a single bet will also be unwilling to make more than one independent bet of the same type.

Erol Peköz is associate professor in the School of Management at Boston University. Thanks are due for helpful comments from Zvi Bodie, Stephen Ross, Paul Samuelson, and an anonymous referee.

[1] Ross, 1999.

Nielsen (1985) found necessary and sufficient conditions for a gambler with a concave utility function to eventually accept a sequence of bounded good bets, and Lippman and Mamer (1988) extended this to unbounded, identically distributed bets. Recently Ross (1999) extended this to independent but nonidentically distributed bets. The essential idea given in Lippman and Mamer (1988) and Ross (1999) is that if the utility function decreases faster than exponentially in the negative direction, the small risk of a loss can be magnified enough to overwhelm the benefits of a gain even for arbitrarily long sequences of good bets. Gollier (1996) gave some related results on how the availability of future optional bets can increase the attractiveness of a current bet, but the eventual attractiveness of a sufficiently large number of optional good bets has not been directly studied. See Bodie (1995) for a discussion of related phenomena surrounding long-term stock market investing.

Here the author shows what happens when the gambler has the option to quit early: A sufficiently long sequence of good bets should always be accepted, meaning that given a sequence of positive mean bets, for sufficiently large n you should always agree to sequentially make the first n of them with the option to quit early. This holds provided the gambler's utility function is not bounded from above, the expected utility of a single bet is finite, and a condition on the bet means and variances holds.

This result does not hold in the setting of Ross (1999) without a stopping option. There the total number of bets to be made is viewed as fixed in advance, while in the setting herein it is viewed as variable up to some maximum number that is fixed in advance. Note that the gambler is not allowed to play as long as it takes to get ahead, but is only allowed a maximum of n bets, which must be fixed in advance.

It is interesting to note that the strategy of quitting when you reach some large wealth level does not perform well. This is because even with arbitrarily long sequences of good bets, there can always be some small chance that the game ends with a very large loss, and a utility function can always be found that magnifies this loss more than enough to make the game unacceptable. One uses the strategy of quitting whenever the gambler's wealth goes below the starting wealth, and for sufficiently long sequences of good bets, the benefit of large gains always eventually overwhelms the risk of losses.

As a final note, much controversy exists over Samuelson's fallacy and the behavioral issues surrounding it. Benartzi and Thaler (1999), for example, used it as an example illustrating the limitations of expected utility theory in explaining behavior. This article shows that if faced with the opportunity to play a long sequence of favorable bets with the option to quit early, accepting the game is rational from an expected utility point of view.

The organization of the article is as follows. The next section contains the main result. The "Multiplicative Gambles" section has an analogous result for multiplicative payoffs. The "Summary" section summarizes the conclusions, and the "Appendix" section provides the proofs.

Main Result

Consider an infinite sequence of available bets, and let S_n be the wealth of a gambler who makes the first n bets. Ross (1999) defines a utility function U to have the Eventual

Acceptance Property (EAP) if

$$E[U(S_n)] > U(0) \quad \text{for some } n > 0,$$

meaning for some n the gambler will be willing to make the first n bets. Expanding on this notion, say that the utility function has the Eventual Acceptance with Stopping Option Property (EASOP) if

$$E[U(S_{T \wedge n})] > U(0) \quad \text{for some } n > 0 \text{ and stopping time } T,$$

where $x \wedge y$ denotes $\min(x,y)$, meaning that for some fixed n the gambler will be willing to make the first n bets with the option to stop early at any time, knowing the outcomes of all previous bets. Note that in the former case the gambler is required to agree to make all n bets in advance, whereas in the latter case the gambler agrees to make up to n sequential bets with the option to stop at any time along the way.

Here the concern is only with sequences of positive mean bets, which are represented as a sequence of independent random variables $X_1, X_2, \ldots$ with means μ_i satisfying $\mu = \inf_i \mu_i > 0$, and first let $S_n = \sum_{i=1}^n X_i$ denote the wealth of a gambler after the nth bet is made. Ross (1999) studied sequences of such positive mean bets and showed, subject to some additional side conditions, that the EAP holds if and only if a concave utility function, U, satisfies

$$\lim_{x \to -\infty} U(x)e^{\gamma x} \to 0$$

for all $\gamma > 0$. This type of condition was also previously given in Lippman and Mamer (1988, Theorem 4). The related side condition requiring finite bet variance is discussed in Lippman and Mamer (1988, Example 2).

More examples where the EAP does not hold can be easily created from this, such as when $U(x) = -e^{-x}$ on $(-\infty, 0)$ and is arbitrary above 0, preserving concavity. Letting, for example, $X_i \sim N(\frac{1}{4}, 1)$, a straightforward computation (see Ross, 1999), shows that the expected utility of the sequence S_n approaches $-\infty$ exponentially in n.

A consequence of the main result below is that the story is different for EASOP: With a sequence of good bets (and some side conditions), essentially all unbounded utility functions satisfy the EASOP. For sufficiently large n a gambler should always be willing to sequentially make the first n bets with the option to stop early. The author now formally states the main result.

Theorem 1. *Let $X_1, X_2, \ldots$ be a sequence of independent random variables with means μ_i and variances σ_i^2, and let the wealth after the nth bet be $S_n = \sum_{i=1}^n X_i$. Suppose*

$$\mu = \inf_i \mu_i > 0 \tag{1}$$

and

$$\sum_{i=1}^{\infty} \sigma_i^2 / i^2 < \infty. \tag{2}$$

If a utility function, U, satisfies

$$\lim_{x \to \infty} U(x) = \infty \tag{3}$$

and

$$\alpha = \inf_i E[U(X_i)] > -\infty, \tag{4}$$

then the EASOP holds.

Proof: See Appendix.

This means that, while for some utility functions it is rational to reject even an arbitrarily large number of sequential good bets, it is usually not rational if you have the option to quit early. This has implications for any sequentially observed investment situations—including, for example, investments made over a lifetime.

The intuition for the result is as follows. For the EAP not to hold, the utility function must decrease faster than an exponential function in the negative direction. This means that when losses are extremely costly, even with an arbitrarily long sequence of good bets the risk of incurring losses overwhelms the benefits of the gains. If early stopping is allowed, the gambler can always stop before losses get too severe, and this cuts the risk far enough to make a sufficiently long sequence of good bets attractive.

Note 1. Condition (2) is satisfied when there is a uniform bound on the variances; i.e., when

$$\sigma_i^2 \leq \sigma, \ \forall i.$$

Note 2. For the proof, use the rule of stopping when wealth first goes below zero. Some condition like (2) is needed for this rule because if the variances increase too quickly, it may be too easy to go below zero even at high wealth levels. For example, suppose

$$U(x) = \begin{cases} \log_2 x & x > 0 \\ -10 & x < 0 \\ 0 & x = 0 \end{cases}$$

and

$$X_i = \begin{cases} +2^i & \text{with probability } 2/3 \\ -2^i & \text{with probability } 1/3. \end{cases}$$

In this case the rule will not work well. Starting with zero, it is easy to see that any time you lose a bet, your total wealth will be negative, and then expected utility equals

$$\log_2(2^{n+1} - 1)(2/3)^n - 10(1 - (2/3)^n),$$

which can be seen to be negative for all $n \geq 1$.

MULTIPLICATIVE GAMBLES

In many situations, gambles are multiplicative rather than additive. The same result as Theorem 1 essentially holds.

Theorem 2. *Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $\mu_i = E[\log(X_i)]$ and $\sigma^2 = \mathrm{Var}(\log(X_i))$, and let the wealth after the nth bet be $S_n = \prod_{i=1}^{n} X_i$. Suppose $\mu = \inf_i \mu_i > 0$ and $\sum_{i=1}^{\infty} \sigma_i^2 / i^2 < \infty$. If a utility function, U, satisfies Equations (3) and (4) above, then the EASOP holds for any such sequence of multiplicative bets.*

Proof: See Appendix.

SUMMARY

While examples of utility functions exist in which very large numbers of good bets should not be accepted, no such utility functions exist if stopping early is allowed. A good stopping strategy for this turns out to be "quit while you're behind" rather than the perhaps more intuitive "quit while you're ahead." This is subject to the condition that utility functions are not bounded from above, any single bet has finite expected utility, and a mild condition on the payoff means and variances holds.

The author also mentions the idea that if someone given the option to make a sufficiently large number of good bets believes he or she has the option to stop early, accepting the game is rational. This may partly explain Samuelson's fallacy, the perception that a large sequence of good bets should always be accepted.

APPENDIX

Proof of Theorem 1: Without loss of generality, assume that the gambler starts at wealth level 0 with $U(0) = 0$. Define the stopping time T so that the gambler makes up to a total of n bets but will quit if the wealth ever goes below zero. Thus

$$T = n \wedge \min\{i \geq 1 : S_i < 0\},$$

where $x \wedge y$ denotes $\min(x, y)$. It will be shown that for sufficiently large n, $E[U(S_T)] > 0$ and hence the EASOP holds.

First, let

$$A = \{S_T > L\}$$

denote the event the gambler's wealth ends above L, for some fixed L. Compute the expected utility of the final wealth by conditioning using

$$E[U(S_T)] = E[U(S_T)|A]P(A) + E[U(S_T)|A^c]P(A^c), \tag{A1}$$

where A^c denotes the complement of A.

Use a slight generalization of the strong law of large numbers, which appears, for example, in Durrett (1996, p. 69, Exercise 8.4), stating that if $X_1, X_2, \ldots$ are independent mean 0 random variables satisfying condition (2) then $S_k / k \to 0$ as $k \to \infty$, where $\to$ denotes convergence with probability 1.

Applying this to the mean-zero variables $(X_i - \mu_i)$ obtains

$$(S_k - \sum_{i=1}^{k} \mu_i)/k \to 0$$

and thus

$$S_k/k \to \sum_{i=1}^{k} \mu_i/k \geq \mu,$$

which implies

$$\forall \epsilon > 0, \exists m : P(S_k \geq k\mu, \forall k > m) \geq 1 - \epsilon. \tag{A2}$$

Let $\epsilon = 1/2$ and use Equation (A2) to find an m so that

$$P(S_k \geq k\mu, \forall k > m) \geq 1/2. \tag{A3}$$

Using this choice of m, let

$$p = \frac{1}{2} \prod_{i=1}^{m} P(X_i \geq \mu), \tag{A4}$$

and note that $E[X_i] \geq \mu$ implies $p > 0$. Then by Equation (3), pick L sufficiently large so that

$$U(L) > -\alpha/p, \tag{A5}$$

and finally pick n sufficiently large so that $n > \max(m, L/\mu)$.

Now, because this choice of n gives $n\mu > L$,

$$\begin{aligned} P(A) &\geq P(S_n > L, S_k \geq k\mu, \forall k \geq 1) \\ &= P(S_k \geq k\mu, \forall k \geq 1) \\ &\geq P(B)P(C|B), \end{aligned}$$

where

$$B = \{X_k \geq \mu, \forall k : 1 \leq k \leq m\}$$

and

$$C = \{S_k \geq k\mu, \forall k > m\}.$$

A straightforward coupling argument gives $P(C|B) \geq P(C)$, and by Equation (A3), $P(C) \geq 1/2$. In addition, clearly $P(B) = 2p$, and combining these, $P(A) \geq p$.

Because U is nondecreasing in x, one must have $E[U(S_T)|A] \geq U(L)$, and condition (4) also gives $E[U(S_T)|A^c] \geq \alpha$. Thus by Equation (A1) we have

$$E[U(S_T)] \geq \alpha + U(L)p > 0, \tag{A6}$$

where the final inequality follows from the choice of L in Equation (A5). Note that for the first inequality of Equation (A6), assume $\alpha \leq 0$; otherwise the theorem would trivially hold with $T = 1$. This establishes the result. ∎

Proof of Theorem 2: Follows by applying Theorem 1 to the logarithms of the variables X_i. ∎

REFERENCES

Benartzi, S., and R. H. Thaler, 1999, Risk Aversion or Myopia? Choices in Repeated Gambles and Retirement Investments, *Management Science*, 45(3): 364-381.

Bodie, Z., 1995, On the Risk of Stocks in the Long Run, *Financial Analysts Journal*, 51(3): 18-22.

Durrett, R., 1996, *Probability: Theory and Examples*, 2nd ed. (Belmont, Calif: Wadsworth Publishing).

Gollier, C., 1996, Repeated Optional Gambles and Risk Aversion, *Management Science*, 42(11): 1524-1530.

Lippman, S. A., and J. W. Mamer, 1988, When Many Wrongs Make a Right, *Probability in the Engineering and Informational Sciences*, 2: 115-127.

Nielsen, L. T., 1985, Attractive Compounds of Unattractive Investments and Gambles, *Scandinavian Journal of Economics*, 87: 463-473.

Pratt, J. W., and R. J. Zeckhauser, 1987, Proper Risk Aversion, *Econometrica*, 55: 143-154.

Ross, S. A., 1999, Adding Risks: Samuelson's Fallacy of Large Numbers Revisited, *Journal of Financial and Quantitative Analysis*, 34(3): 323-339.

Samuelson, P., 1963, Risk and Uncertainty: A Fallacy of Large Numbers, *Scientia*, 98: 108-113.

Samuelson, P., 1989, The $\sqrt{N}$ Law and Repeated Risktaking, in: *Probability, Statistics, and Mathematics, Papers in Honor of Samuel Carlin* (Academic Press, Inc.), 291-306.