

Paintings, Plane Tilings, & Proofs

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One quality of a work of art is that the viewer will often see more in the work than the artist intended. If you are mathematically inclined, no doubt you see some mathematics in the tilings.

Over the centuries many artisans and artists have employed plane tilings in their work. Artisans used tiles for floors and walls because they are durable, waterproof, and beautiful; and artists portrayed realistically the tilings they encountered in the scenes they painted. One quality of a work of art is that the viewer will often see more in the work than the artist intended. If you are mathematically inclined, no doubt you see some mathematics in the tilings. Here are some examples where plane tilings on floors, walls, and in paintings underlie proofs of some well-known (and some not-so-well-known) theorems.

The floor tiling in *Street Musicians at the Doorway of a House* by Jacob Ochtervelt (1634–1682) on the cover of this magazine consists of squares of two different sizes. If one overlays a grid of larger squares, as illustrated in blue in Figure 1a, a proof of the Pythagorean theorem results, one often attributed to Annairizi of Arabia (circa 900). Such a proof is called a “dissection” proof, as it indicates how the squares on the legs of the triangle can be dissected and reassembled to form the square on the hypotenuse. Shifting the blue overlay grid to the position illustrated in Figure 1b yields another dissection proof, one attributed to Henry Perigal (1801–1899). Of course, other positions for the blue grid will yield further

proofs—indeed, there are uncountably many different such dissection proofs of the Pythagorean theorem generated from the floor tiling in Ochtervelt’s painting! No wonder the two-square tiling in Ochtervelt’s painting is sometimes called the Pythagorean tiling.

The floor tiling in the Salon de Carlos V in the Real Alcazar in Seville (Figure 2a) provides the basis for the well-known “Behold!” proof of the Pythagorean theorem ascribed to Bhaskara (12th century) in Figure 2b. However, tilings provide “picture proofs” for many theorems other than the Pythagorean. The tiles in the Salon de Carlos V also illustrate the arithmetic mean-geometric mean inequality, as illustrated in Figure 2c. [Exercise: Change the dimensions of the rectangular tiles in Figure 2c to illustrate the harmonic mean-geometric mean inequality.] Tiling the plane with rectangles of different sizes but in the same general pattern as in the Salon de Carlos V yields a proof of the sine-of-the-sum trigonometric identity (Figure 3). [Exercise: Do similar tilings yield proofs of other trigonometric identities?]

The floor tiling in *A Lady and Two Gentlemen* by Jan Vermeer (1632–1675) appears at first glance to be rather ordinary (Figure 4a). It is, after all, just a version of Cartesian graph paper. But the same blue overlay pattern employed with

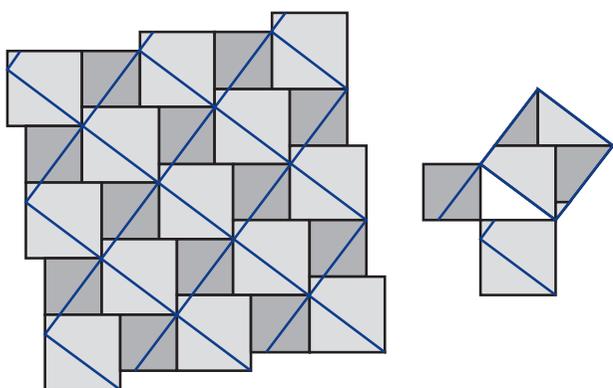


Figure 1a. Annairizi of Arabia’s proof of the Pythagorean Theorem.

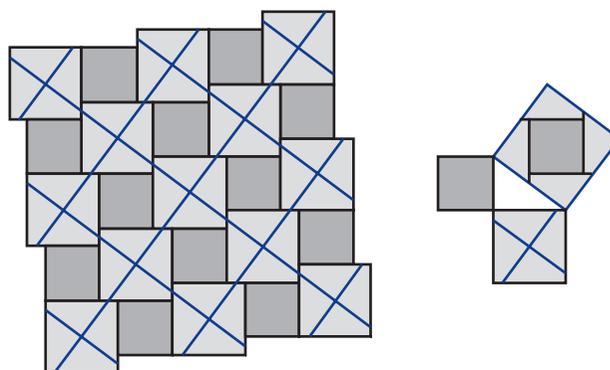


Figure 1b. Henry Perigal’s proof of the Pythagorean Theorem.



Figure 2a. Salon de Carlos V, Real Alcazar, Seville.

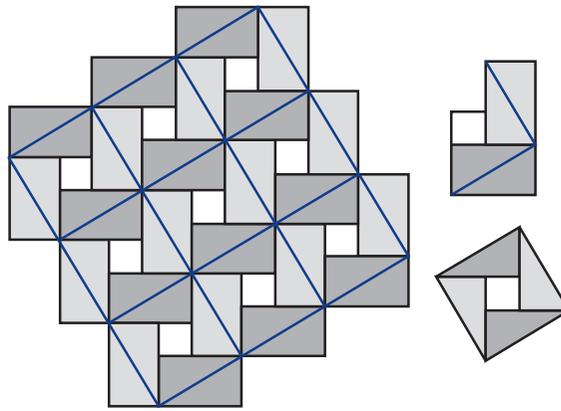
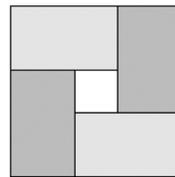


Figure 2b. Bhaskara's proof of the Pythagorean Theorem.



$$4ab \leq (a+b)^2$$

$$\therefore \sqrt{ab} \leq \frac{a+b}{2}$$

Figure 2c. The arithmetic mean—geometric mean inequality.

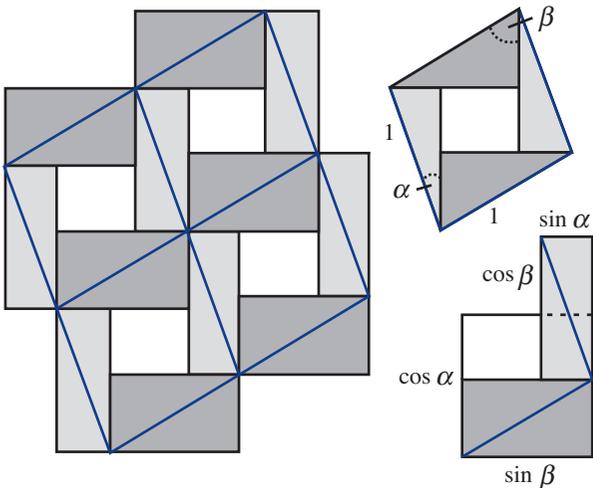


Figure 3. $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$.

Ochtervelt's painting and in the Salon de Carlos V provides a proof (Figure 4b) of

THEOREM 1. If lines from the vertices of a square are drawn to the midpoints of adjacent sides, then the area of the smaller square so produced is one-fifth that of the given square.

A different overlay (the blue circle and squares in Figure 4c) proves

THEOREM 2. A square inscribed in a semicircle has two-fifths the area of a square inscribed in a circle of the same radius.

So far the tilings we've examined have used squares and rectangles—but any quadrilateral will tile the plane, a fact often employed by the Dutch graphic artist M. C. Escher (1898-1972). Figure 5 illustrates the underlying quadrilateral tiling for one of his better known works, *No. 67 (Horsemen)*, and uses such a tiling to prove

THEOREM 3. The area of a convex quadrilateral Q is equal to one-half the area of a parallelogram P whose sides are parallel to and equal in length to the diagonals of Q .

[Exercise: Theorem 3 holds for non-convex quadrilaterals as well—can you prove it with a tiling?]

Just as quadrilaterals tile the plane, so do triangles. Figure 6 illustrates tilings based on equilateral triangles in the Salon de Embajadores in the Real Alcazar in Seville. Of course, an arbitrary triangle will tile the plane, and Figure 7a uses such a tiling with an arbitrary triangle to prove a theorem similar to Theorem 1:

THEOREM 4. If the one-third points on each side of a triangle are joined to opposite vertices, the resulting triangle is equal in area to one-seventh that of the initial triangle.

[Exercise: What happens if you replace the “one-third” points in Theorem 4 with “one- n th” or k/n th?] The same tiling—but with a different overlay—proves (see Figure 7b)

THEOREM 5: The medians of a triangle form a new triangle with three-fourths the area of the original triangle.

We conclude by returning to a rectangular tiling found in *The Courtyard of a House in Delft* (Figure 8a) by Pieter de Hooch



Figure 4a. *A Lady and Two Gentlemen* by Jan Vermeer (1632–1675). Courtesy of Herzog-Anton Ulrich Museum-Kunstmuseum des Landes.

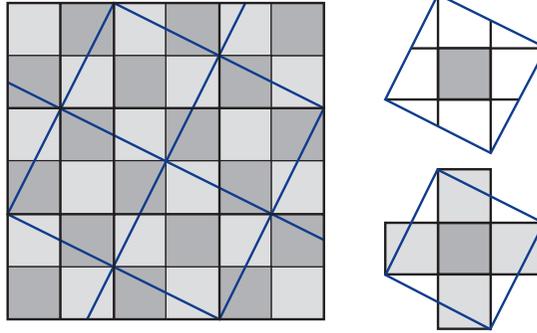


Figure 4b. A proof of Theorem 1.

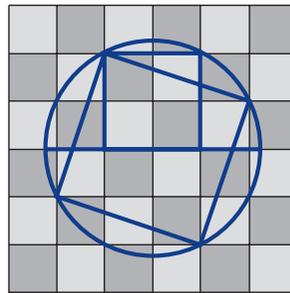


Figure 4c. A proof of Theorem 2. (Note the big square's sidelength.)

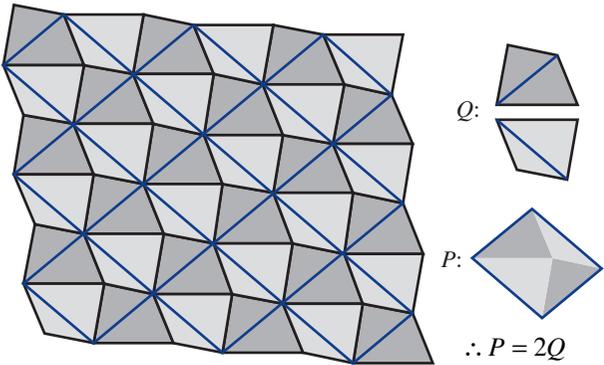


Figure 5. A proof of Theorem 3.

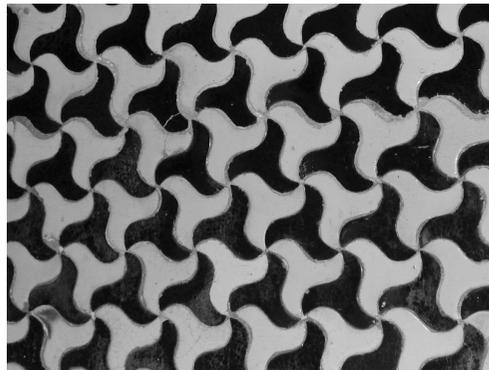


Figure 6. Tiles in the Real Alcazar in Seville.

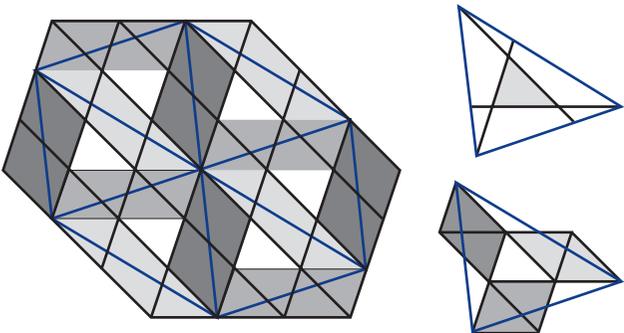


Figure 7a. A proof of Theorem 4.

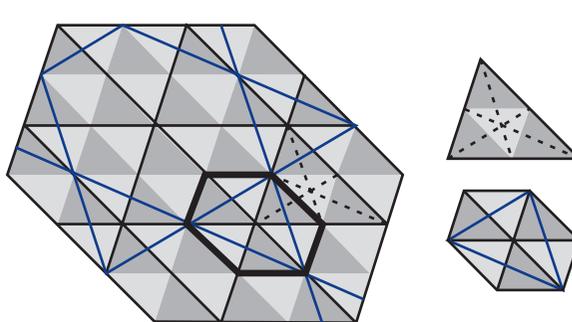


Figure 7b. A proof of Theorem 5.

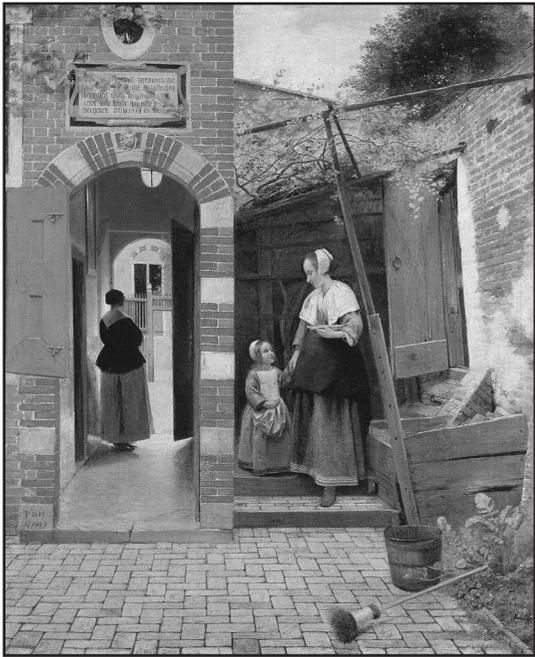


Figure 8a. *The Courtyard of a House in Delft* by Pieter de Hooch (1629–1684). Courtesy of the National Gallery, London.

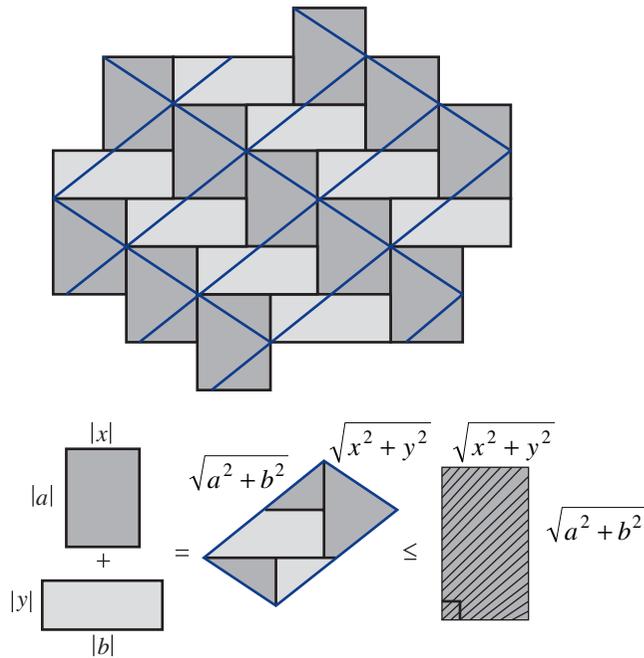


Figure 8b. The Cauchy-Schwarz inequality.

(1629-1684). In the painting, the courtyard is tiled with bricks all the same size, but it is easy to see that bricks of two different sizes could be used, as illustrated in the top part of Figure 8b. With the blue overlay, this tiling forms part of a proof of the Cauchy-Schwarz inequality in two dimensions. No doubt the reader will find other examples of beautiful mathematics—including theorems and proofs—in many other works of art.

For Further Reading

An excellent introduction to tilings is Branko Grünbaum and G. C. Shephard's *Tilings and Patterns: An Introduction*, W. H. Freeman (1989). For an on-line collection of the paintings of Dutch Baroque era painters such as Ochtervelt, Vermeer, and de Hooch, visit the Web Gallery of Art at <http://www.kfki.hu/~arthp>. ■