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General Spacefilling Curve Heuristics and Limit Theory for the Traveling Salesman Problem

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The tour given by the spacefilling curve heuristic applied to a random sample of points from the unit square differs in subtle ways from the shortest tour through those points. In particular, the expected length of the heuristic tour is found to be asymptotic to $\sqrt{n}$ times a periodic function of $\log n$ for a broad class of spacefilling curves. The theory is developed further by detailing the self-similarity properties of the spacefilling curve that are useful for obtaining the asymptotics of the expected length of the heuristic path. Finally, the theory of martingales is applied to obtain tail bounds that yield rigorous almost sure asymptotics for the length of the heuristic tours.

1. INTRODUCTION TO THE SPACEFILLING CURVE HEURISTIC

The basic ingredient of the spacefilling curve heuristic is a surjective mapping ψ: $[0, 1] \to [0, 1]^2$ such that for each $x \in [0, 1]^2$ one can *quickly compute* a $t \in [0, 1]$ such that $\psi(t) = x$. The set of such ψ is large, and any of these ψ can be used to build heuristic methods for many problems of

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geometric combinatorial optimization. The main purpose of this article is to explore how these heuristics behave. For specificity we focus on the traveling salesman problem (TSP), and mainly we consider geometric features of the tours that one obtains on applying the heuristic to a random samples of points from $[0, 1]^2$. Still, there is a general theme that is illustrated by these investigations, and it lives in the assertion that many methods of continuous analysis can be of use in problems of discrete optimization. This theme goes at least a little way to suggest that the classical theory of combinatorial computational complexity may not be as distant from the theory of numerical computational complexity as is commonly supposed.

The essential idea of the spacefilling curve heuristic applied to the TSP is the suggestion that we can find a short tour through a set of n points $\{x_1, x_2, \ldots, x_n\} \subset [0, 1]^2$ by visiting the points in the order of their preimages in $[0, 1]$. More formally, we have a three-step process where we (1) compute a set of points $\{t_1, t_2, \ldots, t_n\} \subset [0, 1]$ such that $\psi(t_i) = x_i$ for each $1 \le i \le n$, (2) order the t_i so that $t_{(1)} \le t_{(2)} \le \cdots \le t_{(n)}$, and, finally, (3) define a permutation σ: $[1, n] \to [1, n]$ by requiring $x_{\sigma(i)} = \psi(t_{(i)})$. The path that visits $\{x_1, x_2, \ldots, x_n\}$ in the order of $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}$ will be called the *spacefilling curve path*, and the tour that closes this path by adding the step from $x_{\sigma(n)}$ back to $x_{\sigma(1)}$ will be called the *spacefilling curve tour*.

For the TSP heuristic built on a spacefilling curve ψ to be effective, one wants ψ to be as smooth as possible. Many of the classical spacefilling curves are Lipschitz of order one-half, which is to say there exists a constant c_ψ so that for any $0 \le s, t \le 1$ one has $\|\psi(s) - \psi(t)\| \le c_\psi |s - t|^{1/2}$. Furthermore, one can easily check that no spacefilling curve can be Lipschitz of order greater than one-half. To see this, just note that the union of the images of $[i/k, (i + 1)/k]$ for $0 \le i < k$ must cover $[0, 1]^2$, so at least one of these images must have diameter at least $2\pi^{-1/2}k^{-1/2}$.

Many of the classic spacefilling curves have a further property that makes them particularly compatible with probabilistic investigations; they are measure preserving in the sense that for any Borel set $A \subset [0, 1]^2$, $\lambda_1(\psi^{-1}(A)) = \lambda_2(A)$, where λ_d denotes the Lebesgue measure on $\mathbb{R}^d$. In a later section, we will illustrate how this property helps one to translate many questions about random samples in $[0, 1]^2$ to simpler questions about a random sample in $[0, 1]$. This stochastic dimension reduction comes unbidden here, but in other contexts it can make light work of problems that otherwise would be perplexing.

As noted in the discussion by Adler (1986), the spacefilling curve heuristic for the TSP dates back at least to unpublished work of S. Kakutani in 1966. Still, the idea has become much better known and better understood since the work of Bartholdi and Platzman (1982, 1988). A particular distinction of these works is that they show that the required preimage calculation can be done effectively and quickly. The works also illustrate that problems other than just the TSP can be addressed by use of the spacefilling curve heuristic, though on this point one should also note the independent developments of the spacefilling curve heuristic in communication theory by Bailey (1969), in matching theory by Imai (1986), and in real analysis by Kahane (1976) and Milne (1980).

Motivation for the present investigation comes in good part from a probabilistic result of Platzman and Bartholdi (1989). Suppose X_i, $1 \le i \le n$, are independent random variables that are uniformly distributed in $[0, 1]^2$, and let $L_n^{\mathrm{SFC}} = L^{\mathrm{SFC}}(X_1, X_2, \ldots, X_n)$ denote the length of a spacefilling heuristic path through $\{X_1, X_2, \ldots, X_n\}$. Platzman and Bartholdi (1989) showed that for a specific ψ, which had been designed to make the heuristic as effective as possible, there are two constants β^+ and β^{-1} such that

$$\beta^+ = \limsup_{n\to\infty} EL_n^{\mathrm{SFC}}/\sqrt{n} \qquad \text{and} \qquad \beta^- = \limsup_{n\to\infty} EL_n^{\mathrm{SFC}}/\sqrt{n},$$

where $\beta^+ - \beta^- > 0$.

Perhaps the most striking aspect of this result is that it offers a sharp contrast to the behavior of the length $L_n^{\mathrm{OPT}} = L_n^{\mathrm{OPT}}(X_1, X_2, \ldots, X_n)$ of the *shortest path* through the random sample $\{X_1, X_2, \ldots, X_n\}$, for which the famous theorem of Beardwood *et al.* (1959) tells us that there is a constant $\beta > 0$ such that for $n \to \infty$ we have

$$L_n^{\mathrm{OPT}}/\sqrt{n} \to \beta,$$

where the convergence takes place in expectation as well as with probability one.

One goal of this article is to develop the limit result of Platzman and Bartholdi a bit further with the twin aims of laying out the explicit properties of ψ that are needed to provide an asymptotic understanding of the expectation of L_n^{SFC} and of rendering as precisely as possible the nature of the oscillatory behavior of $L_n^{\mathrm{SFC}}/\sqrt{n}$. A second goal of this article is to complement the understanding of EL_n^{SFC} with information on the behavior of the tail probabilities $P(|L_n^{\mathrm{SFC}} - EL_n^{\mathrm{SFC}}| \ge t)$.

Bartholdi and Platzman (1989) give few details in their discussion of the almost sure behavior of L_n^{SFC}, but part of the plan that they sketch offers to base the almost sure convergence theory of L_n^{SFC} on the methods of subadditive Euclidean functionals given in Steele (1981). Since there are substantial differences between the geometry of the spacefilling curve heuristic and that of subadditive Euclidean functionals, this approach does not seem to be an easy one. In contrast, there are major benefits to be found in basing the almost sure analysis of L_n^{SFC} on martingale difference methods. By applying martingale methods developed for the analysis of optimal TSP tours, one needs little work to produce strong bounds on the tail probabilities $P(|L_n^{\mathrm{SFC}} - EL_n^{\mathrm{SFC}}| \geq t)$. In turn, these make quick and complete work of the almost sure behavior of L_n^{SFC} while contributing additional understanding of the whole process.

2. Structural Assumptions and Main Results

For *all* of the results considered here we assume our spacefilling curves are measure preserving and Lipschitz of order one-half; but, to obtain a serious asymptotic understanding of EL_n^{SFC} one seems to need to bring out further properties of ψ. The properties described below are found in many of the classical spacefilling curves, and they reflect the important fact that most of the classical spacefilling curves (such as those of Hilbert (1891) or Peano (1890)) have aspects of self-similarity.

A1. *Dilation Property*. There is an integer $p \geq 2$ such that for all $0 \leq s, t \leq 1$,

$$\|\psi(s) - \psi(t)\| = \sqrt{p}\left\|\psi\left(\frac{s}{p}\right) - \psi\left(\frac{t}{p}\right)\right\|.$$

A2. *Translation Property*. For $1 \leq i \leq p$, if $(i-1)/p \leq s, t \leq i/p$, then

$$\|\psi(s) - \psi(t)\| = \|\psi(s + 1/p) - \psi(t + 1/p)\|.$$

A3. *Bimeasure Preserving Property*. Given any Borel set A in $[0, 1]$, one has

$$\lambda_1(A) = \lambda_2(\psi(A)),$$

where λ_d is the Lebesgue measure on $\mathbb{R}^d$.
The main results of this article are the following two theorems.

THEOREM 1. *If a heuristic tour is built using a spacefilling curve ψ that satisfies the Dilation Property* (A1), *the Translation Property* (A2), *and the Bimeasure Preserving Property* (A3), *then there exists a continuous function φ of period of* 1 *such that*

$$\lim_{n\to\infty} \frac{EL_n^{\mathrm{SFC}}}{\sqrt{n}\varphi(\log_p n)} = 1,$$

where p is the integer appearing in assumptions A1 *and* A2.

THEOREM 2. *If a spacefilling curve φ has the Bimeasure Preserving Property* (A3), *then there are constants A and B such that for all $t \geq 0$,*

$$P(|EL_n^{\text{SFC}} - EL_n^{\text{SFC}}| \geq t) \leq B \exp(-At^2/\log t).$$

3. PROOF OF THEOREM 1

We first record a simple lemma that makes explicit the correspondence created by ψ between uniform random variables in $[0, 1]^2$ and $[0, 1]$.

LEMMA 1. *Suppose X is a random variable that is uniformly distributed in* $[0, 1]^2$. *Let ψ^* be a function that for every $x \in [0, 1]^2$ selects a preimage of x, that is, ψ^* satisfies $\psi(\psi^*(x)) = x$. For t defined by $t = \psi^*(X)$, we have that, if the spacefilling curve satisfies the bimeasure preserving assumption* A3, *then t is uniformly distributed in* $[0, 1]$.

The proof of the lemma can be safely omitted, but one should note that there are measure preserving mappings ψ that satisfy A1 and A2 while failing to have A3 and failing to have the correspondence property.

By the lemma, an independent uniform random sample $\{X_1, X_2, \ldots, X_n\} \subset [0, 1]^2$ corresponds under ψ^* to a uniform random sample $\{t_1, t_2, \ldots, t_n\} \subset [0, 1]$. Thus, the length L_n^{SFC} of the path provided by spacefilling heuristic is given by

$$L_n^{\text{SFC}} = \sum_{i=1}^{n-1} \|\psi(t_{(i)}) - \psi(t_{(i+1)})\|,$$

where $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ are the order statistics of $t_1, t_2, \ldots, t_n$. On taking the expectation we find

$$EL_n^{\text{SFC}} = \sum_{i=1}^{n-1} \int_0^1 \int_0^t \|\psi(s) - \psi(t)\| f_i(s, t)\, ds\, dt,$$

where $f_i(s, t)$ is the joint density of $(t_{(i)}, t_{(i+1)})$ given by

$$f_i(s, t) = \frac{n!}{(i-1)!(n-i-1)!} s^{i-1}(1-t)^{n-i-1} I_{[0 \leq s < t \leq 1]}.$$

Here we should remark that this last density formula, as well as those we use subsequently, can be obtained in many ways, through the most general of these seems to be direct integration of the full joint density of $(t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)})$. If we write the basic simplex as $S_n = \{(x_1, x_2, \ldots, x_n)\colon 0 < x_1 < \cdots < x_n < 1\}$, then the fundamental fact is that $(t_{(1)} \leq t_{(2)} \leq$

$\cdots \le t_{(n)})$ has the density $n! I_{S_n}(x_1, x_2, \ldots, x_n)$, and specialized distributional information about individual variables (or pairs of variables) thus can be obtained by direct integration. For illustrations of the use of the uniform distribution on the simplex to obtain the derivation of expressions like that given for $f_i(s, t)$, one can consult Devroye (1986).

The following elementary lemma points out a somewhat surprising formula for the sum of the joint densities of successive order statistics. We will use it to write EL_n^{SFC} in a simpler form.

LEMMA 2. *For* $0 \le s < t \le 1$, *we have*

$$\sum_{i=1}^{n-1} f_i(s, t) = n(n-1)(1-t+s)^{n-2}.$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n-1} f_i(s, t) &= \sum_{i=1}^{n-1} \frac{\partial^2}{\partial s \partial t} P(t_{(i)} \le s,\ t_{(i+1)} \le t) \\
&= \frac{\partial^2}{\partial s \partial t} \sum_{i=1}^{n-1} \int_s^t \int_0^s \frac{n!}{(i-1)!(n-i-1)!} u^{i-1}(1-v)^{n-i-1}\, du\, dv \\
&= \frac{\partial^2}{\partial s \partial t} \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} s^i[(1-s)^{n-i} - (1-t)^{n-i}] \\
&= \frac{\partial^2}{\partial s \partial t} [1 - (1-s)^n - (1-t+s)^n + (1-t)^n] \\
&= n(n-1)(1-t+s)^{n-2}. \quad \blacksquare
\end{aligned}$$

By substituting (1) into the integral representation for EL_n^{SFC} and letting $x = t - s$, we have

$$EL_n^{\mathrm{SFC}} = \int_0^1 \int_0^{1-x} \|\psi(s+x) - \psi(s)\| n(n-1)(1-x)^{n-2}\, ds\, dx.$$

This is a charming exact formula for the expected length of the heuristic tour, but there is an approximate formula that is a bit simpler and more natural for asymptotic analysis.

LEMMA 3.

$$\frac{EL_n^{\mathrm{SFC}}}{\sqrt{n}} = n^{3/2} \int_0^1 \int_0^{1-x} \|\psi(s+x) - \psi(s)\| e^{-nx}\, ds\, dx + o(1).$$

Proof. We have to estimate the absolute value Δ_n of the difference between the principal terms

$$\frac{1}{\sqrt{n}}\int_0^1\int_0^{1-x}\|\psi(s+x)-\psi(s)\|n(n-1)(1-x)^{n-2}\,ds\,dx$$

and

$$n^{3/2}\int_0^1\int_0^{1-x}\|\psi(s+x)-\psi(s)\|e^{-nx}\,ds\,dx.$$

From the completely crude bound $\|\psi(s+x)-\psi(s)\|\le\sqrt{2}$, we see

$$\Delta_n\le\sqrt{2}\int_0^1|n^{1/2}(n-1)(1-x)^{n-2}-n^{3/2}e^{-nx}|dx,$$

and since $|(1-x)^{n-2}-e^{-nx}|\le\{(1-x)^n-(1-x)^{n-2}\}+\{e^{-nx}-(1-x)^n\}$ easy integrations show $\Delta_n=O(n^{-1/2})$, leaving room to spare.

If we define the function $k(x)$ by

$$k(x)=\int_0^{1-x}\|\psi(s+x)-\psi(s)\|ds,$$

then by the usual considerations of Laplace's asymptotic method, the behavior of $EL_n^{\mathrm{SFC}}/\sqrt{n}$ is determined by the behavior of $k(x)$ in the neighborhood of 0, so we may just as well write

$$\frac{EL_n^{\mathrm{SFC}}}{\sqrt{n}}=n^{3/2}\int_0^q k(x)e^{-nx}\,dx+o(1),$$

where $q=1/p$.

We can now generalize an observation from Platzman and Bartholdi (1989) to show there is a function $r(x)$ that captures the scaling properties of ψ, and, at the same time, offers an approximation for the function $k(x)$.

LEMMA 4. *For $q=1/p$, one can define a continuous function r on $[0,\infty)$ with the following properties*:

(a) $r(x)\le c_\psi\sqrt{x}$, $0\le x<\infty$.
(b) $|k(x)-r(x)|\le c_\psi x^{3/2}$, $0\le x\le q$.
(c) $r(x)=\sqrt{p}\,r(x/p)$, $0\le x<\infty$.

Proof. For any $0\le x\le q$, we see from the definition of $k(x)$ that

$$k(x)\ge\sum_{i=1}^{p}\int_{(i-1)q}^{iq-x}\|\psi(s)-\psi(s+x)\|ds. \tag{1}$$

Next, by assumption A2, changing variables, and using assumption A1, we obtain

$$\sum_{i=1}^{p} \int_{(i-1)q}^{iq-x} \|\psi(s) - \psi(s+x)\| ds = p \int_0^{q-x} \|\psi(s) - \psi(s+x)\| ds$$

$$= \int_0^{pq-px} \left\| \psi\left(\frac{s}{p}\right) - \psi\left(\frac{s+px}{p}\right) \right\| ds$$

$$= \frac{1}{\sqrt{p}} \int_0^{1-px} \|\psi(s) - \psi(s+px)\| ds$$

$$= \frac{1}{\sqrt{p}} k(px). \tag{2}$$

By combining (1) and (2), we find the basic fact

$$k(x) \geq \frac{1}{\sqrt{p}} k(px). \tag{3}$$

If we replace x by x/p, in inequality (3), we then have $k(x) \leq \sqrt{p}k(x/p)$ for $0 \leq x \leq 1$. Thus, if we define a sequence of functions $\{f_n(x)\}_{n\geq 0}$ by

$$f_n(x) = p^{n/2} k\left(\frac{x}{p^n}\right), \qquad 0 \leq x \leq p^{n-1},$$

then these functions are pointwise monotonic on ever increasing intervals:

$$f_n(x) \leq f_{n+1}(x) \qquad 0 \leq x \leq p^{n-1}. \tag{4}$$

By the Lipschitz property of ψ, we have $k(u) \leq c_\psi u^{1/2}$ for $0 \leq u \leq 1$, so for $0 \leq x \leq p^{n-1}$ we have

$$f_n(x) \leq p^{(n+1)/2} c_\psi \sqrt{x/p^{n+1}} = c_\psi \sqrt{x}. \tag{5}$$

The last two inequalities show that for all $x \geq 0$ that $\{f_n(x)\}$ is a bounded, monotone sequence. We will denote the limit by $r(x)$ and show that this $r(x)$ satisfies the requirements of the lemma. To begin, we see that letting $n \to \infty$ in (5) already yields requirement (a).

To obtain the other required properties of $r(x)$, we introduce a sequence of functions g_n that we can show will *decrease* to $r(x)$. We let

$$g_0(x) = c_\psi x^{3/2} + k(x),$$

and we note by (2) that for $0 \le x \le q$ we have

$$\begin{aligned} g_0(x) &= \left\{\sum_{i=1}^{p} \int_{(i-1)q}^{iq-x} \|\psi(s) - \psi(s+x)\|ds + \sum_{i=1}^{p-1} \int_{iq-x}^{iq} \|\psi(s) - \psi(s+x)\|ds\right\} \\ &\quad + c_\psi x^{3/2} \\ &= \frac{1}{\sqrt{p}} k(px) + \sum_{i=1}^{p-1} \int_{iq-x}^{iq} \|\psi(s) - \psi(s+x)\|ds \\ &\quad + \{c_\psi p x^{3/2} - c_\psi (p-1) x^{3/2}\} \\ &= \frac{1}{\sqrt{p}} k(px) + c_\psi p x^{3/2} + \sum_{1=1}^{p-1} \left\{\int_{iq-x}^{iq} \|\psi(s) - \psi(s+x)\|ds - c_\psi x^{3/2}\right\}. \end{aligned} \tag{6}$$

By the Lipschitz property of ψ, we have

$$\int_{iq-x}^{iq} \|\psi(s) - \psi(s+x)\|ds - c_\psi x^{3/2} \le 0,$$

so after replacing x by x/p, we have for $0 \le x \le 1$ that $g_0(x) \ge \sqrt{p}\, g_0(x/p)$. We rescale g_0 to define $g_n(x) = p^{n/2} g_0(x/p^n)$ and note that our inequality for g_0 yields the monotonicity relation:

$$g_n(x) \ge g_{n+1}(x), \qquad 0 \le x \le p^{n-1}.$$

The definition of g_n tells us

$$g_n(x) - f_n(x) = c_\psi x^{3/2}/p^n, \qquad 0 \le x \le p^{n-1}, \tag{7}$$

so $g_n(x)$ also converges to $r(x)$ as $n \to \infty$. This implies that for $0 \le x \le p^{n-1}$,

$$f_n(x) \le r(x) \le g_n(x). \tag{8}$$

To show the part (b) we just need $n = 0$ to see that for $0 \le x \le q$

$$k(x) = f_0(x) \le r(x) \le g_0(x),$$

and, consequently,

$$|k(x) - r(x)| \le |g_0(x) - f_0(x)| = c_\psi x^{3/2} \qquad 0 \le x \le q.$$

All that is left to finish the proof is to establish (c), but this is almost immediate by the definition of r. We have

$$\begin{aligned} r\left(\frac{x}{p}\right) &= \lim_{n\to\infty} p^{n/2} k\left(\frac{x}{p^{n+1}}\right) \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{p}} \left\{ p^{(n+1)/2} k\left(\frac{x}{p^{n+1}}\right) \right\} \\ &= \frac{r(x)}{\sqrt{p}}, \end{aligned}$$

completing the proof of the lemma.

To bring $r(x)$ into action, we note that by the first two parts of the lemma, we have

$$\begin{aligned} \frac{EL_n^{\text{SFC}}}{\sqrt{n}} &= n^{3/2} \int_0^q r(x) e^{-nx}\, dx + o(1), \\ &= n^{3/2} \int_0^\infty r(x) e^{-nx}\, dx + o(1). \end{aligned} \tag{9}$$

Still, the power of this representation is evident only when we use the third part of the lemma to connect $r(x)$ to a periodic function of $\log_p n$.

LEMMA 5. *There is a continuous function* φ: $[0, \infty) \to [0, \infty)$ *with period* 1 *such that*

$$\varphi(\log_p n) = n^{3/2} \int_0^\infty r(x) e^{-nx}\, dx.$$

Proof. We let $I(n)$ denote the right hand side, divide the interval $[0, \infty)$ into subintervals $[p^k/n, p^{k+1}/n]$, $-\infty < k < \infty$, and the make the change of variables $x = p^{k+u}/n$ on the subinterval $[p^k/n, p^{k+1}/n]$ to get

$$\begin{aligned} I(n) &= n^{3/2} \sum_{k=-\infty}^{\infty} \int_{p^k/n}^{p^{k+1}/n} r(x) e^{-nx}\, dx \\ &= \sqrt{n} \sum_{k=-\infty}^{\infty} \int_0^1 r(p^{k+u}/n) p^{k+u} \exp(-p^{k+u}) \ln p\, du. \end{aligned}$$

By Lemma 4, part (c), we have

$$I(n) = n^{1/2} \sum_{k=-\infty}^{\infty} \int_0^1 r(p^u/n) p^{3k/2+u} \exp(-p^{k+u}) \ln p \, du$$

$$= \ln p \int_0^1 r(p^{u-\log_p n})/\sqrt{p^{u-\log_p n}} \sum_{k=-\infty}^{\infty} p^{(3/2)(k+u)} \exp(-p^{k+u}) \, du.$$

We then define the function φ by

$$\varphi(x) = \ln p \int_0^1 r(p^{-(x-u)})/\sqrt{p^{-(x-u)}} h(u) \, du,$$

where h is defined by

$$h(u) = \sum_{k=-\infty}^{\infty} p^{3/2(k+u)} \exp(-p^{k+u}) \, du.$$

We note that both h and φ are continuous, and by the integral representation for $I(n)$ we have

$$I(n) = \varphi(\log_p n).$$

Next, to check that φ is periodic function with period 1, we just note that $r(x) = \sqrt{p} r(x/p)$ tells us

$$\begin{aligned} \varphi(x+1) &= \ln p \int_0^1 r(p^{-(x+1-u)})/\sqrt{p^{-(x+1-u)}} h(u) \, du \\ &= \ln p \int_0^1 \sqrt{p} r(p^{-(x-u)}/p)/\sqrt{p^{-(x-u)}} h(u) \, du \\ &= \varphi(x). \end{aligned} \tag{10}$$

Finally, we check that φ is bounded from below by a positive constant. To begin we note that by continuity and periodicity there would otherwise be an x_0 such that $\varphi(x_0) = 0$. But by the integral definition of φ this would imply that r vanishes on the interval $[p^{-x_0}, p^{-x_0+1}]$, and this in turn would imply that r is identically zero by the recursive property of r in Lemma 4. Hence we conclude that φ never vanishes and consequently is bounded below by a positive constant.

Dividing both sides of (9) by $\varphi(\log_p n)$ and applying Lemma 5, we have proved that

$$\lim_{n\to\infty} \frac{EL_n^{\text{SFC}}}{\sqrt{n}\varphi(\log_p n)} = 1.$$

4. Proof of Theorem 2

To set up the application of martingale differences, we first let $\mathcal{F}_k$ be the σ-field generated by $\{X_1, X_2, \ldots, X_k\}$, and then define

$$d_i = E(L_n^{\text{SFC}} | \mathcal{F}_i) - E(L_n^{\text{SFC}} | \mathcal{F}_{i-1}). \tag{11}$$

Once can easily check that $\{d_i, 1 \le i \le n\}$, is a martingale difference sequence and that we have the basic representation

$$L_n^{\text{SFC}} - EL_n^{\text{SFC}} = \sum_{i=1}^{n} d_i.$$

This type of representation for the *optimal* tour length L_n was introduced in Rhee and Talagrand (1987), and its application was subsequently refined in Rhee and Talagrand (1989b). One can consult these articles for additional information on the role of martingales in problems such as the TSP; and, for general background on the applications of martingales in combinatorics, one does very well by consulting Chap. 7 of Alon *et al.* (1992).

For the path taken here, we recall two results from Steele (1989). The first is a rather special L^p inequality that requires some machinery from martingale theory for its proof, but the second is just an exercise in applying Markov's inequality.

LEMMA 6. *Let d_i, $1 \le i \le n$, be a martingale difference sequence. If there are two constants c_1 and c_2 such that for $p > 1$ and $1 \le i \le n$ we have*

$$\|d_i\|_\infty \le c_1(n - i + 1)^{-1/2},$$

and

$$\|d_i\|_p \le c_2(p/n)^{1/2},$$

then there is a constant c_3 such that

$$\left\| \sum_{i=1}^{n} d_i \right\|_p \le c_3 p^{1/2}(\log p)^{1/2}.$$

LEMMA 7. *For any random variable Z, there are constants a and b so that*

$$P(|Z| \ge t) \le a \exp(-bt^2/\log t), \qquad t \ge 0,$$

if and only if, there is a constant c such that for all $p \geq 1$ that

$$\|Z\|_p \leq cp^{1/2}(\log p)^{1/2}.$$

The proof of Theorem 2 thus reduces to showing that the martingale differences defined in (11) satisfy the conditions of Lemma 6. To reexpress those conditional expectations in a more convenient form, we introduce independent random variables $\{\tilde{X}_i: 1 \leq i \leq n\}$ with the uniform distribution on $[0, 1]^2$ that are also independent of $\{X_i: 1 \leq i \leq n\}$, and we define $L_n^{\mathrm{SFC}}(i) = L^{\mathrm{SFC}}(X_1, X_2, \ldots, X_{i-1}, \tilde{X}_i, X_{i+1}, \ldots, X_n)$. The purpose of introducing these variables is that we find the new representation

$$d_i = E(L_n^{\mathrm{SFC}} - L_n^{\mathrm{SFC}}(i)|\mathcal{F}_i).$$

Now we need to find bounds on d_i. In sympathy with our earlier notation, we put $\tilde{t}_i = \psi^*(\tilde{X}_i)$, and, for our first crude bound, we use the definition of the heuristic and the Lipschitz property of ψ to find

$$|L_n^{\mathrm{SFC}} - L_n^{\mathrm{SFC}}(i)| \leq 2c_\psi \min_{i<j\leq n} |\tilde{t}_i - t_j|^{1/2} + 2c_\psi \min_{i<j\leq n} |t_i - t_j|^{1/2}.$$

The reason for the restriction on the j's is brought out when we take expectations and use independence to get the bound

$$\begin{aligned} |d_i| = |E(L_n^{\mathrm{SFC}} - L_n^{\mathrm{SFC}}(i)|\mathcal{F}_i)| &\leq E(|L_n^{\mathrm{SFC}} - L_n^{\mathrm{SFC}}(i)||\mathcal{F}_i) \\ &\leq 2c_\psi E(\min_{i<j\leq n} |\tilde{t}_i - t_j|^{1/2}) + 2c_\psi E(\min_{i<j\leq n} |t_i - t_j|^{1/2}|\mathcal{F}_i). \qquad (12) \end{aligned}$$

To complete the computation of these expectations, we note the easy bounds on the tail probabilities

$$P(\min_{i<j\leq n} |\tilde{t}_i - t_j|^{1/2} \geq t) = P(\min_{i<j\leq n} |\tilde{t}_i - t_j| \geq t^2) \leq (1 - t^2)^{n-i},$$

and

$$P(\min_{i<j\leq n} |t_i - t_j|^{1/2} \geq t|\mathcal{F}_i) \leq (1 - t^2)^{n-i}.$$

Since there is a constant α_1 such that for all m,

$$\int_0^1 (1 - t^2)^m \, dt = \frac{1}{2} \pi^{1/2}\Gamma(m + 1)/\Gamma(m + 3/2) \leq \alpha_1(m + 1)^{-1/2},$$

we see there is a constant α_2 such that

$$E \min_{i<j\le n} |\bar{t}_i - t_j|^{1/2} = \int_0^\infty P(\min_{i<j\le n} |\bar{t}_i - t_j|^{1/2} \ge t)\, dt \le \alpha_2(n - i + 1)^{-1/2}$$

and

$$E(\min_{i<j\le n} |t_i - t_j|^{1/2} | \mathcal{F}_i) \le \alpha_2(n - i + 1)^{-1/2}.$$

By applying these two bounds in (12) we see that we have established the first condition of the lemma.

Now, we show that there is constant c_2 such that

$$\|d_i\|_p \le c_2(p/n)^{1/2}.$$

By the definition of d_i and Jensen's inequality, we have

$$E|d_i|^p = E|E(L_n^{\text{SFC}} - L_n^{\text{SFC}}(i)| \mathcal{F}_i)|^p \le E|L_n^{\text{SFC}} - L_n^{\text{SFC}}(i)|^p.$$

For $p \ge 1$ and $x \ge 0$, $y \ge 0$ we have $|x + y|^p \le 2^p(x^p + y^p)$, so

$$E|d_i|^p \le 2^p c_\psi E(\min_{i\ge j\le n} |t_i - t_j|^{p/2}) + 2^p c_\psi E(\min_{j\ne i} |t_i - t_j|^{p/2}).$$

To bound the last two expectations, we note

$$\begin{aligned} E \min_{i\ge j\le n} |\bar{t}_i - t_j|^{p/2} &= \int_0^1 \frac{p}{2}\, t^{p/2} P(\min_{i\ge j\le n} |\bar{t}_i - t_j| \ge t)\, dt \\ &\le \frac{p}{2} \int_0^1 t^{p/2-1}(1 - t)^n\, dt \le \frac{p}{2}\, n^{-p/2}\Gamma(p/2) \\ &\le c(p/n)^{p/2}, \end{aligned}$$

and by a completely parallel argument we also find

$$E\,(\min_{j\ne i} |t_i - t_j|^{p/2}) \le c(p/n)^{p/2};$$

so the second inequality in the hypothesis of Lemma 6 is proved. By applying Lemma 7, we conclude the proof of Theorem 2.

Application to Almost Sure Convergence

By Theorem 1 and Theorem 2 together with the traditional Borel–Cantelli argument, we find the desired strong law:

$$\lim_{n\to\infty} \frac{L_n^{\text{SFC}}}{\sqrt{n}\varphi(\log_p n)} = 1 \qquad \text{a.s.}$$

One should note that Theorem 2 goes much further than is required for this law; and, in most cases, one does better to use Theorem 2 directly, rather than to call on this corollary. Still, because of the natural connection to the Beardwood–Halton–Hammersley theorem and the results of Platzman and Bartholdi (1989), the application deserves to be singled out.

5. Conclusion

There are two open problems that seem particularly compelling.

1. Is there a spacefilling curve ψ for which the corresponding periodic function φ is actually a constant? One may have to give up some of the efficacy of the heuristic tours provided by the ψ, but one would regain the fundamental behavior of the optimal tours as reflected in the Beardwood–Halton–Hammersley theorem. We know from direct computations done in Platzman and Bartholdi (1989) that there are ψ for which φ is nonconstant, but it would be surprising if there were no ψ for which φ is constant. After all, as the works of Salem and Zygmund (1945) and many others show, there are methods for constructing bimeasure preserving, Lipschitz one-half, spacefilling curves that apparently have little to do with the self-similarity properties A1 and A2.

2. Can the tail bound of Theorem 2 be improved to provide a bound of the Gaussian form, where no log t factor is required? That is, can one show that for any Lipschitz one-half, bimeasure preserving, spacefilling curve that one has constants A and B such that for all $t \geq 0$ one has

$$P(|L_n^{\text{SFC}} - EL_n^{\text{SFC}}| \geq t) \leq B \exp(-At^2).$$

We know from the developments of Rhee and Talagrand (1987, 1989a, and particularly 1989b) that the corresponding Gaussian tail bound does hold for the length of the shortest path. Such an inequality for the spacefilling heuristic paths would seem to be easier than that for the optimal paths, but the problem still does not seem to be easy. Apparently, one cannot easily adapt the method of Rhee and Talagrand (1989b) to obtain a Gaussian tail bound for the length of the spacefilling curve heuristic paths.

References

Adler, R. (1986), Commentary, *in* "The Collected Works of S. Kakutani, Vol. II" (R. R. Kalman, Ed.), p. 444, Birkhauser, Boston, 1986.

ALON, N., SPENCER, J. H., AND ERDÖS, P. (1992), "The Probabilistic Method," Wiley–Interscience Series in Discrete Mathematics and Optimization, New York.

BAILEY, T. (1969), Spacefilling curves: Their generation and their application to bandwidth reduction. *IEEE Trans. Inform. Theory* **II-15,** 658–664.

BARTHOLDI, J. J., AND PLATZMAN, L. K. (1982), An $O(n \log n)$ Planar traveling salesman heuristic based on spacefilling heuristic curves, *Oper. Res. Lett.,* 121–125.

BARTHOLDI, J. J., AND PLATZMAN, L. K. (1988), Heuristics based on spacefilling curves for combinatorial problems in Euclidean space, *Management Sci.* **34**(3), 291–305.

BEARDWOOD, J., HALTON, J. H., AND HAMMERSLEY, J. M. (1959), The shortest path through many points, *Proc. Cambridge Philos. Soc.* **55,** 299–327.

DEVROYE, L. (1986), "Non-Uniform Random Variate Generation," pp. 17–19, Springer-Verlag, New York.

HILBERT, D. (1891), Über die Stetige Abbildung einer Linie auf ein Flächenstück, *Math. Ann.* **38,** 459–460.

IMAI, H. (1986), Worst case analysis for planar matching and tour heuristics with bucketing techniques and spacefilling curves, *J. Oper. Res. Soc. Japan* **29**(1), 43–67.

KAHANE, J.-P. (1976), Mesure et dimensions, *in* "Turbulence and Navier Stokes Equations" (A. Dold and B. Eckmann, Eds.), Lecture Notes in Mathematics, Vol. 565, Springer-Verlag, New York.

MILNE, S. C. (1980), Peano curves and smoothness of functions, *Adv. Math.* **35,** 129–157.

PEANO, G. (1890) Sur une courrbe qui remplit toute une aire plane, *Math. Ann.* **36,** 157–160.

PLATZMAN, L. K., AND BARTHOLDI, J. J. (1989), Spacefilling curves and the planar traveling salesman problem, *J. Assoc. Comput. Mach.* **36**(4), 719–737.

RHEE, R. T., AND TALAGRAND, M. (1987), Martingale inequalities and NP-complete problems, *Math. Oper. Res.* **12,** 177–181.

RHEE, R. T., AND TALAGRAND, M. (1989a), Martingale inequalities, interpolation and NP-complete problems, *Math. Oper. Res.* **14,** 91–96.

RHEE, R. T., AND TALAGRAND, M. (1989b), A sharp deviation for the stochastic traveling salesman problem, *Ann. Probab.* **17,** 1–8.

SALEM, R., AND ZYGMUND, A. (1945), Lacunary power series and Peano curves, *Duke Math. J.* **12,** 569–578.

STEELE, J. M. (1981), Subadditive Euclidean functionals and non-linear growth in geometric probability, *Ann. Probab.* **9,** 365–376.

STEELE, J. M. (1989), Seedlings in the theory of shortest paths, *in* "Disorder in Physical Systems: A Volume in Honor of J. M. Hammersley" (G. Grimmett and D. Welsh, Eds.), Cambridge Univ. Press, Cambridge, MA.