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Moving Averages of Ergodic Processes

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Abstract: A necessary and sufficient condition for the almost everywhere convergence of the "moving" ergodic averages $(\phi(n))^{-1} \sum_{i=n-\phi(n)+1}^{n} \chi_E(T^i x)$ is given. The result is then generalized to ergodic flows, and finally constrasted with earlier results of *Pfaffelhuber* and *Jain*.

1. Introduction

The ergodic theorem of Birkhoff states that for an invertible measure preserving transformation of the measure space (X, F, μ) the sequence

$$n^{-1} \sum_{i=1}^{n} f(T^i x) \tag{1}$$

converges a.e. for all $f \in L_1(X, \mathsf{F}, \mu)$. A natural direction for generalization of the ergodic theorem is via a more general averaging process than (1). In particular one has the basic question:

What are the necessary and sufficient conditions on the matrix (a_{ni}) so that the sequence $f_n(x) = \Sigma\, a_{ni} f(T^i x)$ converges a.e. for each $f \in L_1$.

The present work tackles only a special case of the basic question where definitive results can be provided. In particular we consider the matrix (a_{ni}) defined by

$$a_{ni} = \begin{cases} 1/\phi(n) & n - \phi(n) < i \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $\phi(n)$ is a positive non-decreasing function on the integers. For this choice of (a_{ni}) we provide necessary and sufficient conditions for the a.e. convergence of the $f_n(x)$.

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This particular choice of (a_{ni}) is motivated, first of all, by its relevance to the general problem as pointed out in *Akcoglu* and *del Junco* [1975] (see also *Belley* [1974]). The more direct purpose of (2) is, of course, to study the average of the last $\phi(n)$ of the values $f(x), f(Tx), f(T^2x), \ldots, f(T^nx)$. With the motivation of studying such averages one can just as well consider

$$b_{ni} = \begin{cases} \frac{1}{n} & f(n) \leqslant i \leqslant n + f(n) - 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and this study has been carried out by *Pfaffelhuber* [1975] for ergodic transformations and by *Jain* [1975] for independent, Banach space valued random variables.

In the second section of this paper we prove our main result, Theorem 1, which provides the desired necessary and sufficient conditions for the ergodic theorem under the averaging process (2). In the same section we extend the result of Theorem 1 to the physically interesting case of measure preserving flows.

The third section applies Theorem 1 to the counterexample to *Belley*'s conjecture and contrasts the present results to those of *Pfaffelhuber* [1975] and *Jain* [1975]. The fourth section contains a result which contrasts two related summability methods of the type (2) and (3) respectively.

2. Main Results

The basic result of the present paper is the following:

Theorem 1. Let $0 < \phi(n) \leqslant n$ be a non-decreasing integer valued function and suppose T is an ergodic measure preserving transformation on a non-atomic measure space. The sequence

$$\phi(n)^{-1} \sum_{i=n-\phi(n)+1} \chi_E(T^i x) \tag{4}$$

converges a. e. for all $E \in \mathrm{F}$ if and only if

$$\lim_{n \to \infty} \phi(n) / n = c > 0. \tag{5}$$

Moreover, if ξ is the limit a.e. of the sequence (4) then $\xi = \mu(E)$ a.e.

Proof. We will first suppose that $c = 0$ and proceed to construct a set $E \in \mathrm{F}$ for which (4) fails to converge on a set B of measure 1. By our hypothesis (5) we can select a subsequence n_j such that $\sum_{j=1}^{\infty} \phi(n_j) / n_j < 1/2$. Next given a sequence ϵ_j with $\epsilon_j \downarrow 0$ we can apply Rohlin's theorem (see *Halmos* [1956], Theorem 8.1) to obtain for each j a set $F_j \in$ F such that $T^{-i}F_j$ are disjoint for $i = 0, 1, 2, \ldots, n_j$ and such that $\mu \bigcup_{i=0}^{n_j} T^{-i}F_j \geq 1 - \epsilon_j$. We now let

$$E_j = \bigcup_{i=0}^{\phi(n_j)-1} T^{-i}F_j \text{ and } B_j = \bigcup_{i=\phi(n_j)}^{n_j} T^{-i}F_j.$$

One notes that

$$\mu E_j \leq \phi(n_j) / n_j \text{ so setting } E = \bigcup_{j=1}^{\infty} E_j$$

we have $\mu E \leq 1/2$. Also we have

$$\mu B_j \geq (1 - \epsilon_j)(n_j - \phi(n_j)) / (n_j + 1) \text{ so}$$

for $B = \limsup B_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} B_j$ it follows that $\mu(B) = 1$.

Finally we come to the crucial observation. If $x \in B_j$ then there is an n with $\phi(n_j) \leq n \leq n_j$ such that $T^n x, T^{n-1}x, \ldots, T^{n-\phi(n)+1}x$ are elements of $\bigcup_{i=0}^{\phi(n)} T^{-i}F_j$. Since $\phi(n) \leq \phi(n_j)$ this implies that $T^n x, T^{n-1}x, \ldots, T^{n-\phi(n)+1}x$ are elements of $E_j \subset E$. We have thus established that for $x \in B_j$, there is an $n \geq \phi(n_j)$ so that

$$\frac{1}{\phi(n)} \sum_{i=n-\phi(n)+1}^{n} \chi_E(T^i x) = 1. \tag{6}$$

Now if $\phi(n)$ is unbounded (6) implies that for each $x \in B$ we have,

$$\limsup_{n\to\infty} \frac{1}{\phi(n)} \sum_{i=n-\phi(n)}^{n} \chi_E(T^i x) = 1. \tag{7}$$

But, by the ergodicity of T, if the series (6) converges on a set P of positive measure then the limit of (4) must be equal to $\mu(E) < 1$ a.e. on P.

We have thus established the first half of the theorem under the assumption that $\phi(n)$ is unbounded.

To deal with the case when $\phi(n)$ is bounded we note that $\phi(n)$ must be constant, say equal to k, for all $n \geq n_0$ for some n_0. Then by Rohlin's theorem one can choose a set E such that the function $f(x)$ defined by $f(x) = k^{-1} \sum_{i=0}^{k-1} \chi_E(T^i x)$ is larger than 3/4 on a set of positive measure and less than 1/4 on a set of positive measure. Now to show (1) fails to converge we note that for $n \geqslant n_0$

$$\phi(n)^{-1} \sum_{i=n-\phi(n)+1}^{n} \chi_E(T^i x) = \phi(n_0)^{-1} \sum_{i=n-\phi(n_0)+1} \chi_E(T^i x)$$

$$= k^{-1} \sum_{i=n-k+1}^{n} \chi_E(T^i x)$$

$$= f(T^{n+k-1}x).$$

But since T is ergodic one has by the *Poincaré* recurrence theorem that

$$\limsup_{n\to\infty} f(T^{n-k+1}x) \geqslant 3/4 \text{ a.e.}$$

and

$$\liminf_{n\to\infty} f(T^{n-k+1}x) \leqslant 1/4 \text{ a.e.}$$

Consequently the sequence (1) fails to converge a.e.

To prove the second half of the theorem, we may assume by symmetry that

$$\overline{\lim_{n\to\infty}} \frac{1}{\phi(n)} \sum_{i=n-\phi(n)+1} \chi_E(T^i x) \geqslant \mu(E) + \delta \tag{8}$$

for some $\delta > 0$ and a.e. x. By the ergodic theorem one naturally has

$$\lim_{n\to\infty} (n - \phi(n) - 1)^{-1} \sum_{i=1}^{n-\phi(n)-1} \chi_E(T^i x) = \mu(E) \tag{9}$$

for a.e. x. Moreover for fixed x we can choose a subsequence n_j so that (8) and (9) hold along n_j and such that $\phi(n_j)/n_j$ converges to d, $d \geqslant c \neq 0$. But inequalities (8) and (9) immediatly show that

$$\lim_{n_j\to\infty} \frac{1}{n_j} \sum_{i=1}^{n_j} \chi_E(T^i x) \geqslant d(\mu(E) + \delta) + (1-d)\mu(E). \tag{10}$$

This shows that for a.e. x one has

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \chi_E(T^i x) \geqslant \mu(E) + \delta c \tag{11}$$

and this is in contradiction to the ergodic theorem. We have thus proved that for $c \neq 0$ we have a.e. convergence in (4).

The first application of Theorem 1 will be to show that a completely analogous result holds for ergodic flows.

Theorem 2. Let $0 < \phi(t) < 1$ be a non-decreasing real valued function, and suppose T_t is an ergodic, measure preserving flow on a non-atomic Lebesque space X. Then as $t \to \infty$

$$\frac{1}{\phi(t)} \int_{t-\phi(t)}^{t} \chi_E(T_t x)\, dt \tag{12}$$

converges for a.e. x and all $E \in \mathsf{F}$ if and only if

$$\lim_{t \to \infty} \phi(t) / t = c > 0. \tag{13}$$

Proof. If $f(x)$ is a real valued function on [0, 1] we can define a flow on $\{(x, y) : 0 < x < 1, 0 \leqslant y < f(x)\}$ by allowing the point (x, y) to move vertically at unit speed until $(x, f(x))$ and then jump to $(Sx, 0)$ where S is an ergodic transformation of [0, 1]. According to the *Ambrose* and *Kakutani* [1942] representation theorem, any ergodic, measure preserving flow on a Lebesgue space is isomorphic to a flow "built under a function" as just described. Further by *Rudolf*'s Theorem [1975], the function can be assumed to be a step function $f(x)$ which takes on only values α and β where $0 < \alpha < \beta$.

We can thus assume that T_t is a flow built under a step function $f(x)$ as above. Now by Theorem 1 we can construct a set $E \subset [0, 1]$ such that E has measure less than δ and such that

$$\overline{\lim_{n \to \infty}} \frac{1}{[\phi(n)]} \sum_{n-[\phi(n)]+1}^{n} \chi_E(S^i x) = 1 \tag{14}$$

for a.e. $x \in [0, 1]$. Now let $E' = \{(x, y) : x \in E, 0 \leqslant y < f(x)\}$

If $\phi(t) \to \infty$ as $t \to \infty$, we easily obtain

$$\overline{\lim_{t \to \infty}} \frac{1}{\phi(t)} \int_{t-\phi(t)}^{t} \chi_{E'}(T_t z) \geqslant \alpha \tag{15}$$

for a.e. $z \in \{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y < f(x)\}$. By the ergodicity of T_t if (12) converges on a set P of positive measure then the limit must be less than the measure of E' for a.e. $z \in P$. Since δ can be chosen so that $\delta\beta < \alpha$, we have by (15) that (12) cannot converge on any set of positive measure.

To complete the proof we observe that if $\phi(t)$ is bounded, Theorem 2 is proved essentially as in Theorem 1. Also the sufficiency of (13) for convergence of (12) is proved almost without change from the proof of Theorem 1.

3. An Application of the Main Result

As was noted in the introduction, one would certainly like to know conditions on (a_{ij}) such that $f_n(x) = \sum_{i=1}^{\infty} a_{ni} f(T^i x)$ converges a.e. On the basis of spectral considerations the following conjecture has been advanced:

If $p_n(z) = \Sigma\, a_{ni} z^{-i}$ are uniformly bounded and pointwise convergent on the unit circle then f_n converges a.e.

An attempt to prove this conjecture was made by *Belley* [1974] but the question was settled by *Akcoglu* and *del Junco* [1975] where the coice

$$a_{ni} = \begin{cases} \dfrac{1}{[\sqrt{n}]+1} & n \leqslant i \leqslant [\sqrt{n}] + n \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

was shown to be a counter-example to the conjecture.

We note here that for the choice (a_{ni}) given in (2) we have $p_n(z) = \dfrac{1}{\phi(n)} \sum_{i=n-\phi(n)-1}^{n} z^{-i}$.

These functions are also bounded uniformly by 1 on $|z| = 1$ and converge pointwise to the function on $|z| = 1$ which is 1 at $z = 1$ and 0 for $z \neq 1$. Hence we have that the class of matrices (a_{ni}) given by (2) provides an uncountable class of counter examples to the conjecture quoted above.

4. Related Results

As we have already mentioned the summability method (3) represents an average over a moving block as well as (2). If f in (3) is taken to be an increasing positive function defined for all real $x > 0$, the $\phi(n)$ which gives the corresponding method in (2) is $f^{-1}(n)$, or more precisely $[f^{-1}(n)]$, since $f^{l1}(n)$ is not an integer in general. Thus we have the two sequences

$$S_n = \frac{1}{n} \sum_{i=f(n)}^{f(n)+n-1} \chi_E(T^i x), \qquad \bar{S}_n = \frac{1}{\phi(n)} \sum_{i=n}^{n+\phi(n)-1} \chi_E(T^i x) \tag{17}$$

and one might suppose that S_n converges a.e. if and only if $\bar{S}_n$ converges a.e. In fact S_n is a subsequence of $\bar{S}_n$, so the convergence of $\bar{S}_n$ implies the convergence of S_n. However the converse is not true as can easily be seen from the following result by taking T to be the appropriate Bernoulli shift.

Theorem 3. Suppose X_i, $i = 1, 2, \ldots$ are i.i.d. random variables taking values 0 and 1 with probability $\frac{1}{2}$. Let $f(n) = 2^{2^n}$ and $\phi(n) = [\log_2 \log_2 n]$. Then

$$\overline{\lim_{n\to\infty}} \frac{1}{\phi(n)} \sum_{i=n}^{n+\phi(n)-1} X_i = 1 \text{ and } \underline{\lim} \frac{1}{\phi(n)} \sum_{i=n}^{n+\phi(n)-1} X_i = 0, \tag{18}$$

but

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=f(n)}^{n+f(n)-1} X_i = \frac{1}{2}.$$

Proof: Let $n_k = 2^k$ and note that

$$P\{X_i = 1, n_k < i < n_k + \phi(n_k) - 1\} = 2^{-\phi(n_k)} \geqslant \frac{1}{k}.$$

Furthermore since $n_k + \phi(n_k) - 1 < n_{k+1}$, the events

$$A_k = \{X_i = 1, n_k \leqslant i \leqslant n_k + \phi(n_k) - 1\}$$

are independent. Thus by the *Borel-Cantelli* lemma for independent events one has $P(A_k \text{ i.o.}) = 1$, which shows that the first equality in (17) holds. The second equality of (17) is proved identically. To prove the third equality note that

$$P\{|\frac{n}{2} - \sum_{i=f(n)}^{f(n)+n-1} X_i| \geqslant \in n\} = P\{|\frac{n}{2} - \sum_{i=1}^{n} X_i| \geqslant \in n\} = p_n$$

Since the p_n are summable by the usual estimates (e.g. *Bahador, Rao* [1960]) one has the third equality by the (unrestricted) Borel-Cantelli lemma.

A most notable virtue of the summability method (b_{ni}) of (3) is the number of occasions it produces convergence. This is reflected in a small way by (17), and more powerfully by the result of *Jain* [1975], where simple necessary and sufficient conditions are given for independent Banach space valued random variables to converge a.e. under (b_{ni}).

The possibility that the functions $X_i = \chi_E(T^i x)$ constructed in Theorem 1 are independent cannot be ruled out a priori. This would, in fact, correspond to the situation where T is a Benoulli shift with respect to the partition (E, E^c). Nevertheless, we have the following fact.

Theorem 4. If $\phi(n)/\log n \to \infty$ as $n \to \infty$ and $\frac{1}{\phi(n)} \sum_{i=n-\phi(n)+1}^{n} \chi_E(T^i x)$ fails to converge a.e., then the functions $X_i = \chi_E(T^i x)$ cannot be independent.

Proof. We let $Y_i = X_i - \mu(E)$ and note Y_i are bounded with mean 0. We will suppose for now that the X_i are independent. By elementary estimates of the binomial distribution, we have

$$\mu\left(\left|\sum_{i=n-\phi(n)+1}^{n} Y_i\right| \geqslant \epsilon\,\phi(n)\right) < Ce^{-\alpha\phi(n)} \tag{19}$$

for some constants C and $\alpha > 0$, (for even more precise estimates see (*Bahadur, Rao* [1960] or *Cramer* [1938]). Now, easy estimates and the *Borel-Cantelli* lemma show that $\frac{1}{\phi(n)} \sum_{i=n-\phi(n)+1}^{n} \chi_E(T^i x)$ converges.

One reason for noting this property of the $\chi_E(T^i x)$ is to contrast the present method with that of *Pfaffelhuber* [1975], where the following is proved:

If $\phi(n+1) - \phi(n) \geqslant n$ for infinitely many n then there is a measure space (Ω, F, p) and an ergodic, measure preserving transformation T sucht that $\Sigma\, b_{ni} f(T^i x)$ converges at most on a set of p measure 0.

This result was obtained by taking T so that $f(T^k x)$ are independent with finite mean and infinite variance. The particularly simple choice $\phi(n) = n^2$ will thus provide an example where the $X_i = \chi_E(T^i x)$ of Theorem 1 cannot be taken independent and the procedure of *Pfaffelhuber* [1975] be to take the $X_i = f(T^i x)$ independent.

5. A Valuable Problem

One is more often confronted with a plethora of problems than a paucity. We are fortunate in the present circumstance to be able to pinpoint a single problem of particular significance.

Quite without hesitation one can now point to the value of providing necessary and sufficient conditions for the a.e. convergence of $\Sigma\, a_{ni} f(T^i x)$. The results given here, and contrasted to earlier ones, show that this problem contains cases of variety and interest. In fact, the purpose of this report is well served, if greater attention is brought to focus on this central open problem.

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