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AN OPTIMAL BETTING STRATEGY FOR REPEATED GAMES

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Abstract

We consider the problem of finding a betting strategy for an infinite sequence of wagers where the optimality criterion is the minimization of the expected exit time of wealth from an interval. We add the side constraint that the right boundary is hit first with at least some specified probability. The optimal strategy is derived for a diffusion approximation.

GAMBLING SYSTEMS; KELLY CRITERIA; OPTIMALITY CRITERIA; DIFFUSION APPROXIMATION; REPEATED GAMES

1. Introduction

Suppose a gambler is faced with an infinite sequence of identical wagers, and seeks a betting strategy. We propose the optimality criterion of minimizing the expected first-exit time of his wealth from a specified interval subject to the requirement that the probability of reaching the upper boundary before the lower boundary is at least p. The optimal strategy is derived for a diffusion approximation to the original problem.

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of bounded, i.i.d. random variables, with $\mu = EX_1 > 0$, $\sigma^2 = \operatorname{Var} X_1$. A bet of t on the nth trial returns $t + tX_n$. Let $\beta = (\beta_n)_{n=1}^{\infty}$ be a betting strategy where β_n is the proportion of wealth bet on the nth trial. Assume that β_n is a non-negative Borel-measurable function of $(X_1, X_2, \cdots, X_{n-1})$ with $1 + \beta_n X_n \geqq 0$ a.s. For each β, define the associated wealth process $W^{\beta} = (W_n^{\beta})_{n=0}^{\infty}$ where

$$W_0^{\beta} = 1,$$

$$W_n^{\beta} = \prod_{i=1}^{n} (1 + \beta_i X_i), \qquad n \geqq 1.$$

An appealing and well-studied strategy proposed by Kelly (1956) and studied

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by Breiman (1961), Ferguson (1965), Finklestein and Whitley (1981), Ethier and Tavaré (1983) and others is to bet a fixed proportion β^* of wealth on each trial where β^* maximizes $E\log(1+\beta X_1)$. The associated wealth process W^{β^*} has many desirable properties including that $\{W_n^{\beta}/W_n^{\beta^*}, n \geqq 0\}$ is a supermartingale, where β is any other strategy.

For μ small, then under fairly general conditions $\beta^* \approx \mu/\sigma^2$. A complete discussion of this point is given in Lemma 3.1 of Ethier and Tavaré (1983). In this case, the Kelly betting system is much like a proportional betting system (proportional to W_n, μ and $(\sigma^2)^{-1}$) where the constant of proportionality is 1. A generalization is to set $\beta_n = f\mu/\sigma^2$ for all n where f is a positive constant. The incentive to choose $f < 1$ is to reduce the probability of a 'bad event' which could be defined to be the wealth ever dropping below a specified level. The motivation behind our suggested optimality criterion is similar. The proportional betting strategy is discussed in Friedman (1981), Wong (1981) and Gottlieb (1985).

Our problem is as follows: given a choice of $a < 1$, $b > 1$ and $p \in [0,1]$, with $T^\beta = \inf\{t > 0: W_n^\beta \notin (a,b)\}$, find the strategy β which minimizes $E(T^\beta)$ subject to the condition that $P(W_{T^\beta}^\beta \geqq b) \geqq p$.

Note that many games satisfy $\sigma^2 \gg \mu$, with μ small and positive. The pass line at craps (from the casino perspective) has $\mu = 0.014$ and $\sigma^2 \approx 1$. For the case where $\sigma^2 \gg \mu$, a diffusion approximation is appropriate. The above problem is solved for the diffusion approximation to W.

Assume that

(1.1) $\beta_{n+1}(X_1, \cdots, X_n) = f(W_n^\beta), \qquad n \geqq 1$, f some continuous function.

Then,

$$E(W_{n+1}^\beta - W_n^\beta \mid W_0^\beta, \cdots, W_n^\beta) = f(W_n^\beta)W_n^\beta\mu, \tag{1.2}$$

$$\operatorname{Var}(W_{n+1}^\beta - W_n^\beta \mid W_0^\beta, \cdots, W_n^\beta) = f^2(W_n^\beta)(W_n^\beta)^2\sigma^2. \tag{1.3}$$

Following (1.2) and (1.3), define the diffusion process $\{Y^f(t), t \geqq 0\}$ satisfying

$$Y^f(0) = 1$$

$$dY^f = f(Y^f(t))\mu Y^f(t)dt + \sigma f(Y^f(t))Y^f(t)dB(t).$$

Take $\{Y^f(t), t \geqq 0\}$ to be a diffusion approximation to $\{W_n^\beta, n \geqq 0\}$. Here, one time unit of Y^f corresponds to one wager in the W^β process. Let $\tau^f = \inf\{t \geqq 0: Y^f(t) = a \text{ or } b\}$. For $p \in [0,1]$, let

$$C(p) = \{f:(a,b) \to R : f \text{ continuous}, P(\tau^f < \infty) = 1, P(Y^f(\tau^f) = b) = p\},$$

$$C^+(p) = C(p) \cap \{f : f(x) \geqq 0, \text{ all } x \in (a,b)\}.$$

We solve the following problem.

Problem I. If $C^+(p) \neq \varnothing$, find the $f \in C^+(p)$ which minimizes $E(\tau^f)$ over $C^+(p)$. Find the set of p for which $C^+(p) = \varnothing$.

We show that if $C^+(p) \neq \varnothing$, minimizing $E(\tau^f)$ over $C^+(p)$ and over $\bigcup_{q \geq p} C^+(q)$ yield the same answer.

First consider a related problem. Let g be any function with $g(a) = 0$, $g(b) = 1$. Let $E_x(\cdot) = E(\cdot \mid Y^f(0) = x)$ with $E(\cdot) = E_t(\cdot)$ and define P_x and P similarly.

Let

$$u(x) = \sup_{f \text{ continuous}} E_x[g(Y^f(\tau^f) - \lambda \tau^f]. \tag{1.4}$$

Problem II. Find the 'control' f for which the supremum of the right-hand side of (1.4) is attained.

Lemma 1. Suppose that for a given $\lambda > 0$, f is the optimal control for Problem II. Further, suppose that $f \in C^+(p)$. Then f is the optimal control for Problem I.

Proof. Immediate.

Note that the inclusion of non-Markovian, non-anticipatory controls in Problem II would be irrelevant in the sense that the optimal control (assuming existence) over the larger set could be taken to be Markovian. By Lemma 1, the same holds for Problem I, hence (1.1) is innocuous.

2. Derivation of optimal policy

Problem II is solved in this section. Its solution is used to find the optimal control for Problem I. The $\inf_{f \in C^+(p)} E(\tau^f)$ is computed as a function of a, b and p.

Lemma 2. $u''(x) < 0$ for $x \in (a, b)$.

Proof. Choose an $x \in (a, b)$ and $\varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \subset (a, b)$.

Let $\omega(\nu) = n/\nu$. Let F be the class of continuous controls with the following restriction. Let $\theta = \inf\{t > 0 : Y^\omega = x - \varepsilon \text{ or } x + \varepsilon\}$. Require that for $f \in F$, if $Y^f(0) = x$, then for $t \leqq \theta$, $f(\nu) = \omega(\nu)$.

Then,

$$\begin{aligned} u(x) &\geqq \sup_{f \in F} E_x[g(Y^f(\tau^f)) - \lambda \tau^f] \\ &= -\lambda E_x \theta + \sup_{f \text{ continuous}} E_x E_{Y^\omega(\theta)}[g(Y^f(\tau^f)) - \lambda \tau^f] \qquad (2.1) \\ &= -\lambda E_x \theta + E_x u(Y^\omega(\theta)). \end{aligned}$$

For $t \leqq \theta$, Y^{n} is Brownian motion with drift $n\mu$ and instantaneous variance $n^2\sigma^2$.

From Karlin and Taylor (1981), p. 205,

$$P_x(Y^n(\theta) = x + \varepsilon) = \frac{\exp(-2n\mu x/n^2\sigma^2) - \exp(-2n\mu(x-\varepsilon)/n^2\sigma^2)}{\exp(-2n\mu(x+\varepsilon)/n^2\sigma^2) - \exp(-2n\mu(x-\varepsilon)/n^2\sigma^2)}$$

$$= \frac{\dfrac{2\mu\varepsilon}{n\sigma^2} + \dfrac{2\mu^2\varepsilon^2}{n^2\sigma^4} + O\left(\dfrac{\varepsilon^3}{n^3}\right)}{\dfrac{4\mu\varepsilon}{n\sigma^2} + O\left(\dfrac{\varepsilon^3}{n^3}\right)} \tag{2.2}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\mu\varepsilon}{n\sigma^2} + O\left(\frac{\varepsilon^2}{n^2}\right).$$

Further,

$$E_x(\theta) = \frac{\varepsilon^2}{\sigma^2 n^2} + o\left(\frac{\varepsilon^2}{n^2}\right) = O\left(\frac{\varepsilon^2}{n^2}\right). \tag{2.3}$$

From (2.1), (2.2) and (2.3), one gets

$$u(x) \geqq \tfrac{1}{2}u(x+\varepsilon) + \tfrac{1}{2}u(x-\varepsilon) + \frac{1}{2}\frac{\mu\varepsilon}{n\sigma^2}(u(x+\varepsilon) - u(x-\varepsilon)) + O\left(\frac{\varepsilon^2}{n^2}\right).$$

This reduces to

$$\frac{u(x+\varepsilon) - 2u(x) + u(x-\varepsilon)}{\varepsilon^2} \leqq \frac{\mu}{n\sigma^2}\frac{u(x-\varepsilon) - u(x+\varepsilon)}{\varepsilon} + O\left(\frac{1}{n^2}\right).$$

Letting $\varepsilon \to 0$ yields

$$u''(x) \leqq -\frac{2\mu}{n\sigma^2}u'(x) + O\left(\frac{1}{n^2}\right).$$

Noting that $u'(x) > 0$, and letting n be sufficiently large proves that $u''(x) < 0$.

Theorem 1. $f(x) = (\mu/\sigma^2)(1 + \rho x^{-1})$ is the optimal control for Problem II where ρ is some constant.

Proof. For each $x \in (a, b)$, $u(x)$ satisfies the equation

$$+\lambda = \sup_{y \in \mathbb{R}} \{\tfrac{1}{2}\sigma^2 y^2 x^2 u''(x) + (\mu y x)u'(x)\}. \tag{2.4}$$

Maximizing the right-hand side of (2.4) as a function of y yields

$$y = -\frac{\mu}{\sigma^2 x}\left(\frac{u'(x)}{u''(x)}\right). \tag{2.5}$$

From Lemma 2, we know that $u''(x) \neq 0$, $x \in (a, b)$.

Substituting the derived value of y into (2.4) yields

$$\tfrac{1}{2}\sigma^2 x^2 \frac{\mu^2}{\sigma^4 x^2}\left(\frac{u'(x)}{u''(x)}\right)^2 u''(x) + \mu x\left(-\frac{\mu}{\sigma^2 x}\frac{u'(x)}{u''(x)}\right)u'(x) = \lambda. \tag{2.6}$$

Letting $\lambda' = -2\lambda\sigma^2/\mu^2$, (2.6) reduces to

$$\lambda' u''(x) = (u'(x))^2. \tag{2.7}$$

The solution to (2.7) is

$$u(x) = -\lambda' \ln(x+\rho) + k_1, \quad \text{where } \rho \text{ and } k_1 \text{ are constants.}$$

So,

$$u'(x) = \frac{-\lambda'}{x+\rho}, \tag{2.8}$$

$$u''(x) = \frac{\lambda'}{(x+\rho)^2}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.5), and setting $f(x) = y$ gives

$$f(x) = \frac{\mu}{\sigma^2 x}(x+\rho) = \frac{\mu}{\sigma^2}(1+\rho x^{-1}). \tag{2.10}$$

Set

$$p_0 = \frac{1-a}{b-a}.$$

Theorem 2. For $p \in (p_0, 1]$, $f(x) = (\mu/\sigma^2)(1+\rho x^{-1})$, where

$$\rho = \frac{b(a-1) - p(a-b)}{p(a-b) - (a-1)}$$

is the optimal control for Problem I. For $p \leqq p_0$, $C^+(p) = \varnothing$.

Proof. Choose a $p \in (p_0, 1]$. The idea is to find a ρ so that $f(x) \in C^+(p)$. Then, by Lemma 1, the result follows.

Choose a ρ and set

$$f(x) = \frac{\mu}{\sigma^2}(1+\rho x^{-1}). \tag{2.11}$$

For f given in (2.11), Y^f is a diffusion with scale density

$$s(x) = (x+\rho)^{-2}.$$

Let $R(x) = P_x(Y^f_{\tau_f} = b)$. From Karlin and Taylor (1981), Equation (3.10), p. 195,

$$R(x)=\frac{S(x)-S(a)}{S(b)-S(a)}$$

where

$$S(x)=\int^{x} s(\eta)d\eta.$$

So,

$$R(x)=\frac{(b+\rho)(x-a)}{(b-a)(x+\rho)}.$$

Setting $R(1)=p$ and solving for ρ yields

$$\rho=\frac{b(1-a)-p(b-a)}{p(b-a)-(1-a)}. \tag{2.12}$$

Fix ρ using (2.12) and note that for f defined by (2.11), $f\in C(p)$. It remains to consider the sign of $1+\rho x^{-1}$ for $x\in(a,b)$.

For $p\in(p_0,bp_0)$, $\rho\geqq 0$ so $f\in C^{+}(p)$. Note that as $p\downarrow p_0$, $\rho\uparrow\infty$.

For $p\in(bp_0,1)$, $\rho<0$, but $1+\rho x^{-1}\geqq 0$ for $x\in[a,b]$. The last inequality is strict unless $p=1$ and $x=a$.

To verify the last assertion, choose any $f\in C^{+}(p)$ and let $w(x)=x$.

Applying Dynkin's lemma to the submartingale Y^{f} yields

$$Ew(Y^{f}(\tau^{f}))=1+E\int_0^{\tau^{f}}\mu Y^{f}(t)f(Y^{f}(t))dt. \tag{2.13}$$

The left-hand side of (2.13) is $pb+(1-p)a$, while the right-hand side of (2.13) is strictly greater than 1. Hence, $p>p_0$.

Lemma 3. For $p>p_0$,

$$\min_{f\in C^{+}(p)} E(\tau^{f})=\min_{1\geqq q\geqq p}\ \min_{f\in C^{+}(q)} E(\tau^{f}).$$

Proof. Choose p_1, p_2 with $1\geqq p_2>p_1>p_0$ and let $f_1(f_2)$ be the optimal control for Problem I where $p=p_1(p_2)$.

Suppose that $E(\tau^{f_2})<E(\tau^{f_1})$. Let $f^{n}(x)\equiv n$ and let $p_{(n)}=P(Y^{f^{n}}_{\tau^{f^{n}}}=b)$, $e_n=E(\tau^{f^{n}})$. Now, $e_n\to 0$ as $n\to\infty$, so $p_{(n)}\to p_0$ as $n\to\infty$.

Define a sequence $(r_n)_{n=1}^{\infty}$ satisfying $r_n p_{(n)}+(1-r_n)p_2=p_1$. For n large enough, $0<r_n<1$.

Consider the random control u^{n} which uses the control f_2 throughout with probability $1-r_n$ and uses the control $f^{(n)}$ throughout with probability r_n. For n large enough, the probability of first hitting b is p_1, while the expected hitting time to a or b is less than $E(\tau^{f_1})$. Because of the remarks following Lemma 1, this is a contradiction which proves the lemma.

Henceforth, fix $p \in (p_0, 1]$, define ρ by (2.12) and set $f(x) = (\mu/\sigma^2)(1+\rho x^{-1})$. Let $M(x,p) = E_x[\tau^f]$.

Theorem 3. For $p \in (p_0, 1)$,

$$M(1,p) = \frac{2\sigma^2}{\mu^2}\left\{-\ln(1+\rho) + \frac{c}{1+\rho} + d\right\} \tag{2.14}$$

where

$$c = \frac{\ln\left(\frac{a+\rho}{b+\rho}\right)(a+\rho)(b+\rho)}{b-a},$$

$$d = \ln(b+\rho) - \frac{c}{b+\rho}.$$

Proof. Write $M(x)$ for $M(x,p)$.
From Karlin and Taylor (1981), p. 196,

$$\begin{aligned} M(x) &= 2R(x)\int_x^b (S(b) - S(\zeta))m(\zeta)d\zeta \\ &\quad + 2(1-R(x))\int_a^x (S(\zeta) - S(a))m(\zeta)d\zeta \end{aligned} \tag{2.15}$$

where R and S are defined in Theorem 2 and $m(x)$ is the speed density of Y^f. So,

$$\begin{aligned} m(x) &= \left(\frac{\mu^2}{\sigma^2}(x+\rho)^2\right)^{-1}(s(x))^{-1} \\ &= \frac{\sigma^2}{\mu^2}. \end{aligned} \tag{2.16}$$

Substituting (2.16) into (2.15) yields

$$\begin{aligned} M &= 2\left\{\frac{b+\rho}{b-a}\frac{x-a}{x+\rho}\int_x^b \left(\frac{1}{b+\rho} - \frac{1}{\zeta+\rho}\right)\frac{\sigma^2}{\mu^2}d\zeta\right. \\ &\quad \left. + \left(1 - \frac{b+\rho}{b-a}\frac{x-a}{x+\rho}\right)\int_a^x \left(\frac{1}{\zeta+\rho} - \frac{1}{a+\rho}\right)\frac{\sigma^2}{\mu^2}d\zeta\right\} \\ &= \frac{2\sigma^2}{\mu^2}\left\{-\ln(x+\rho) + \frac{(b+\rho)(x-a)\ln(b+\rho)}{(b-a)(x+\rho)}\right. \\ &\quad \left. + \frac{(b-x)(a+\rho)\ln(a+\rho)}{(b-a)(x+\rho)}\right\}. \end{aligned} \tag{2.17}$$

Setting $x = 1$ in (2.17) reduces it to (2.14).

Corollary 1.

$$M(1,1) = \frac{2\sigma^2}{\mu^2}\left\{\ln\left(\frac{b-a}{1-a}\right)\right\}. \tag{2.18}$$

Proof. Set $p = 1$, so $\rho = -a$ and $\{a\}$ is a natural boundary for Y^f. So, letting $\tau^f_{r,s} = \inf\{t > 0 : Y^f(t) = r \text{ or } s\}$,

$$M(x) = \lim_{\delta \downarrow 0} E_x \tau^f_{a+\delta, b}. \tag{2.19}$$

To evaluate $M(x)$, it suffices to replace a in (2.17) with $a + \delta$ and to replace ρ with $-a$ and calculate the limit as $\delta \downarrow 0$.

So,

$$M(x) = -\frac{2\sigma^2}{\mu^2} \ln\left(\frac{x-a}{b-a}\right).$$

Setting x to 1 completes the proof.

We now compare the expectation of the hitting time under the policy given by Theorem 1 to a fixed proportional strategy. For each $\gamma > 0$, let $v^\gamma(x) = \gamma\mu/\sigma^2$, $x \in (a, b)$ and call v^γ a fixed proportional strategy. Let $\gamma^* = \inf\{t > 0 : Y^{v^\gamma} = a \text{ or } b\}$.

It is easy to show that

$$P_x(Y^{v^\gamma}(\gamma^*) = b) = \frac{x^h - a^h}{b^h - a^h}, \tag{2.20}$$

for $x \in (a, b)$ where

$$h = -\frac{2}{\gamma} + 1 \tag{2.21}$$

and

$$E_x(\gamma^*) = \frac{2\sigma^2}{\mu^2 \gamma(\gamma - 2)}\{x^h \ln a/b + a^h \ln b/x + b^h \ln x/a\}, \qquad \gamma \neq 2. \tag{2.22}$$

Various values of a and b are chosen for Tables 2.1 and 2.2. In each table, μ^2/σ^2 is set to 1.

TABLE 2.1
($a = \frac{1}{2}, b = 2$)

p	$M(1,p)$	$\bar{M}(1,p)$	$\frac{\bar{M}(1,p)}{M(1,p)}$
1/3	0	0	
0.4	0.0194	0.0204	1.0515
0.5	0.1178	0.1201	1.0195
0.6	0.2967	0.2977	1.0034
0.7	0.5596	0.5602	1.0011
0.8	0.9192	0.9357	1.0180
0.9	1.4084	1.5210	1.0800
0.95	1.7309	2.0223	1.1684
0.99	2.0714	2.9826	1.4399
0.999	2.1848	4.1731	1.9100
1	2.1972	∞	∞

TABLE 2.2
($a = \frac{1}{4}, b = 2$)

p	$M(1,p)$	$\tilde{M}(1,p)$	$\frac{\tilde{M}(1,p)}{M(1,p)}$
3/7	0	0	
0.5	0.0206	0.0232	1.1262
0.6	0.1184	0.1274	1.0760
0.7	0.3003	0.3106	1.0343
0.8	0.5787	0.5819	1.0055
0.9	0.9870	0.9920	1.0051
0.95	1.2688	1.3166	1.0377
0.99	1.5769	1.8662	1.1835
0.999	1.6755	2.4745	1.4769
1	1.6946	∞	∞

For each p, γ is chosen so that $(1 - a^h)/(b^h - a^h) = p$, where h satisfies (2.21). Then, $E(\gamma^*)$ is computed using (2.22). Call this number $\tilde{M}(1,p)$.

The reader should note that as $p \uparrow 1$, $\gamma \downarrow 0$ and $E(\gamma^*) \uparrow \infty$.

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