Modeling the Yield Curve

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Plan for Talk

- Background
  - What is the yield curve?
  - What makes it interesting and important?
  - Examples
    - Cash
    - Commodities (primarily crude oil)
- Data analysis is simpler with prices
  - How to recognize arbitrage?
  - More natural structure
- Commodities are different
  - Linkages among products

Background

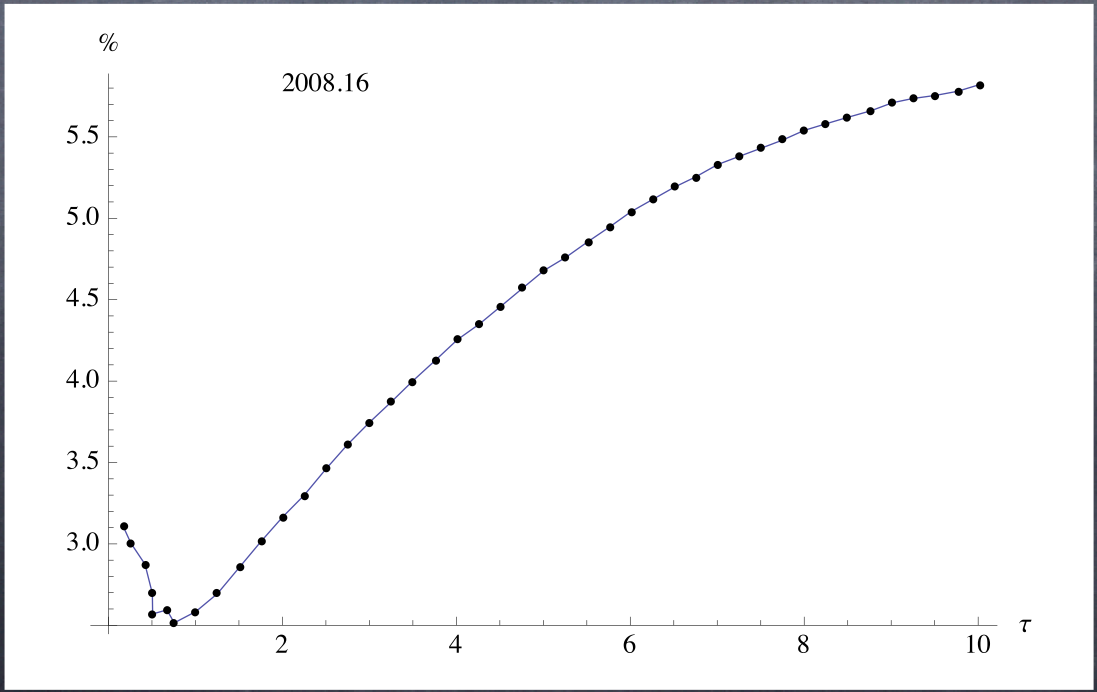
What is the yield curve?

- Interest rates earned on treasury bills/bonds of different maturity (7 Nov 08).
- Curve on a given day is formed by connecting the rates over different maturities.

Term	Interest
3 month	0.269%
6 month	0.84%
2 year	1.32%
5 year	2.56%
10 year	3.79%
30 year	4.27%

What is the yield curve?

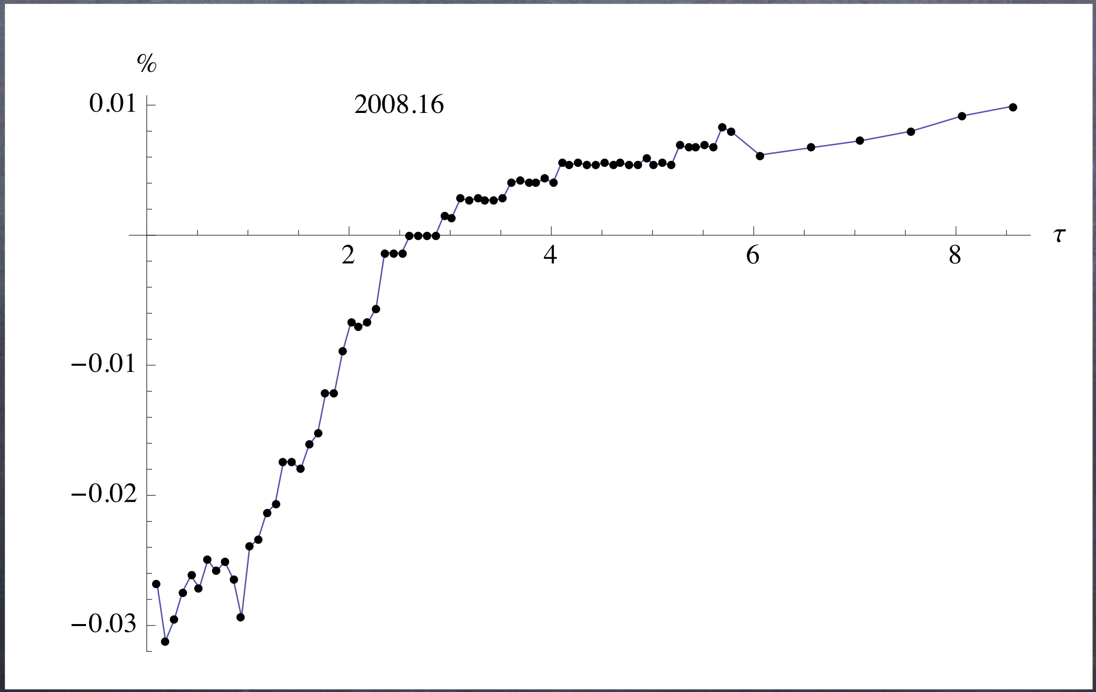
- Interest rates derived from contracts in the Eurodollar option market.
  - London Interbank Offered Rate (LIBOR)
  - Interest rate on a 3-month contract in the <u>future</u>
    - Plot shows instantaneous forward rates.



Many more contracts

Yields for Commodities

- Interest rates implied by the prices of contracts for delivery of a commodity at some future date.
- Light crude oil, same date as prior slide



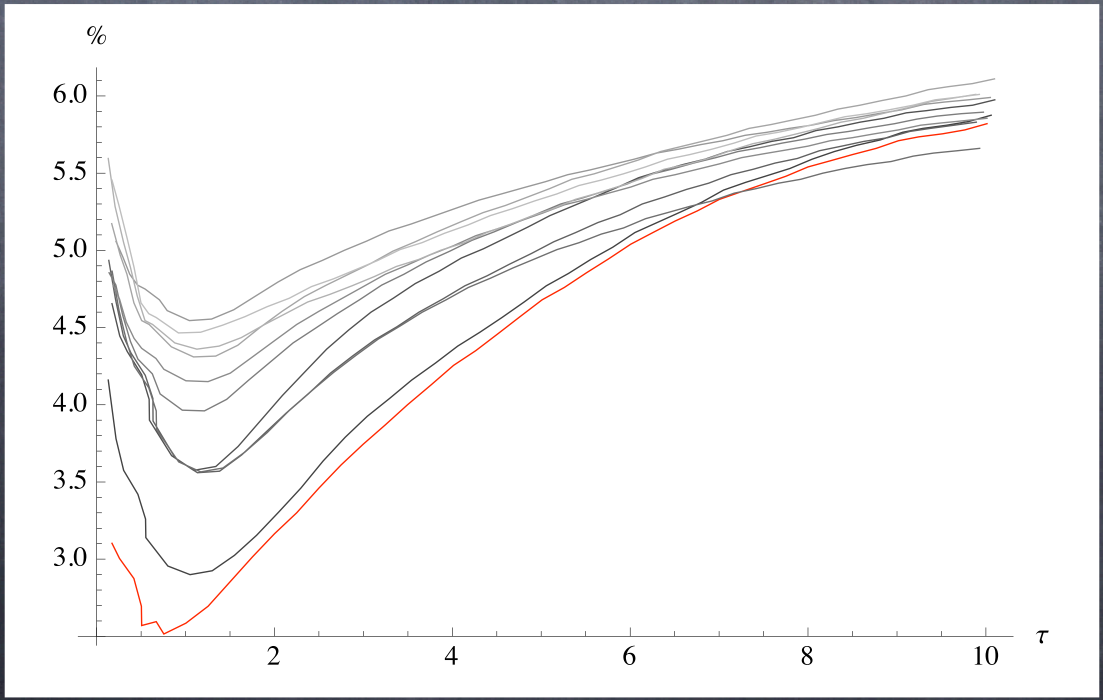
Questions

- What are the dynamics of the yield curve?
- How fast does it change? Can you predict where it's headed?
- How is the yield for cash related to yields implied by commodities that include convenience factors?
- How are yields for various products related to one another?
- What's the connection between these "curves" and the underlying data?

Dynamics

Plots: Cash

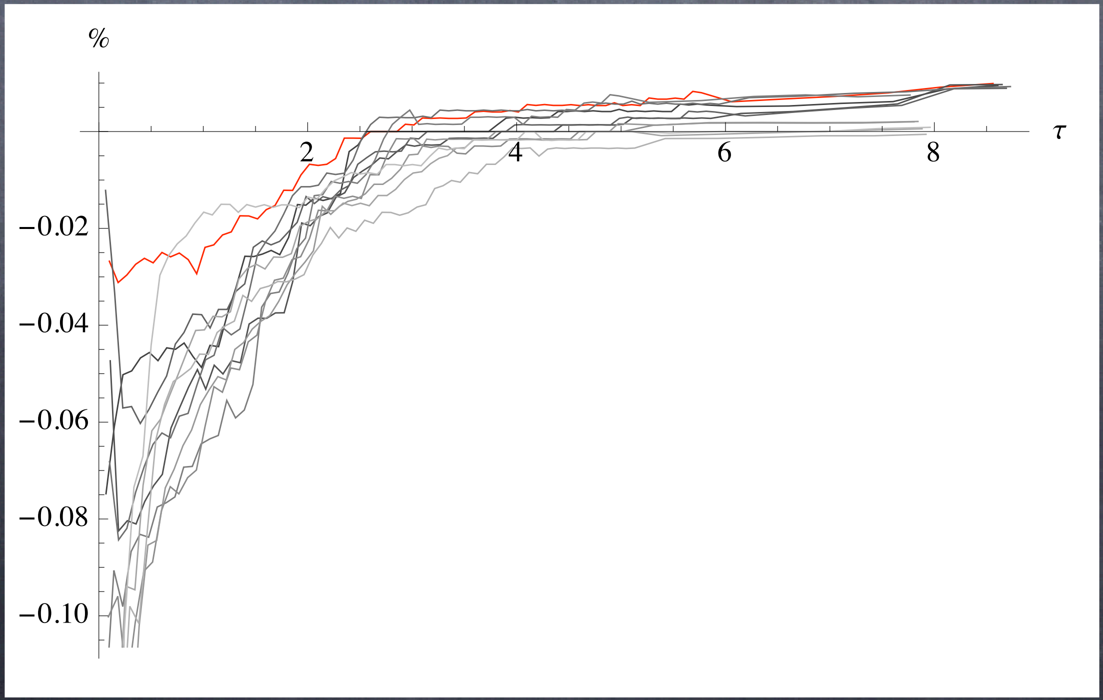
- Yields on cash over period of about 100 days
  - Red curve is the original
  - Gray curves are separated by 10 trading days
- Some changes are large, other curves cluster together



Plots: Light Crude

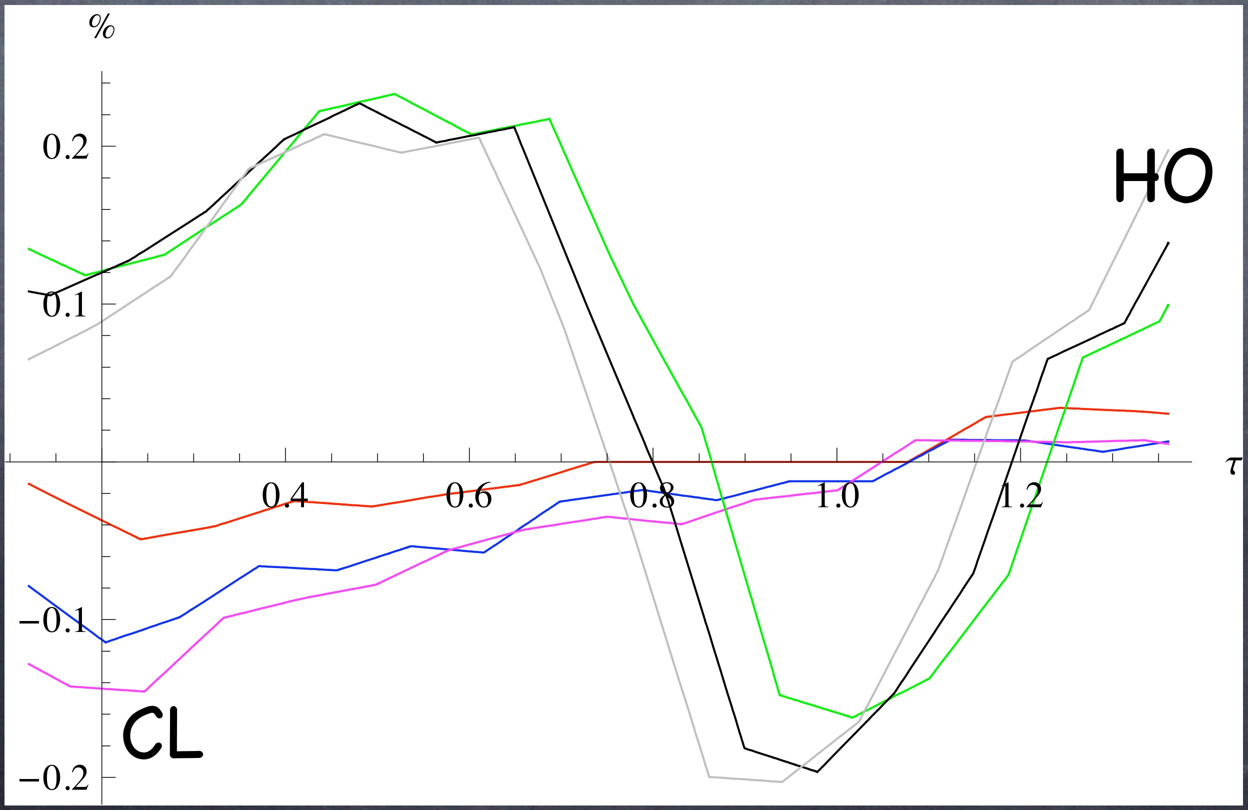
- Yields on crude over same 100 days
  - Red curve is the original, gray 10 days apart
- Rather different appearance from cash, with much less smoothness

Surface plots are also entertaining



Relations Among Products

- Dependent movements over 10-day intervals for heating oil and light crude.
- Strong seasonal pattern for heating oil that is not apparent in yield on light crude.



Models

- Models provide parsimonious way to predict where the curve is heading.
  - Rather than have to predict a “curve”, forecast the value of certain parameters in a regression-like formula for the curve
- Model used resembles a polynomial, but on a logarithmic scale more suited to description of rates.
- Modeling issues (see paper)
  - How many polynomial terms?
  - Does the model allow arbitrage?

Decomposition

- Decompose the yield curve $y_t(\tau)$ into three components

$$y_t(\tau) = \underline{U(\tau)} + \underline{M_t(\tau)} + D_t(\tau)$$

- Long-term unconditional expectation

$$E_s\ y_s(\tau) = U(\tau)$$

- Other terms are separable in τ and t, factoring as $M_t(\tau) = m_t\ g(\tau)$
- Maturity specific component is mean reverting
  - $m(t)$ follows log normal SDE with expectation

$$E_s\ m_t = m_s\ e^{-k(t-s)} \qquad s<t$$

Decomposition, cntd

- Date specific term is also mean reverting, but captures effects that move toward origin with time

  $$E_s\, D_t(\tau) = D_s(\tau+t-s) \qquad s<t,\tau$$

  Each contract carries the date-specific effect

- If d_t follows log normal SDE, then $h(\tau) = e^{-k\tau}$

  $$E_s\, D_t(\tau) = (d_s\, e^{-k(t-s)})\, e^{-k\tau} = D_s(\tau+t-s)$$

- Example (2,2,3)
  - 2^{nd} order unconditional curve
  - 2 maturity-specific functions

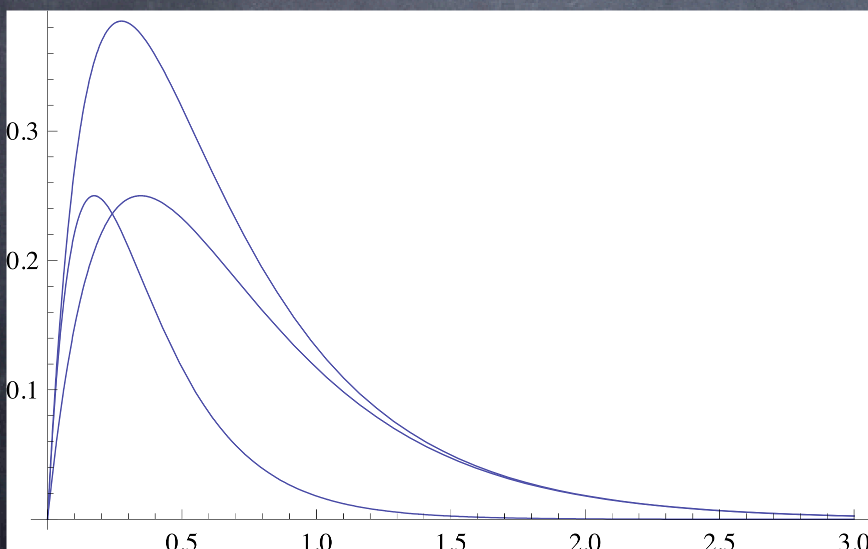
    $$m_{t,1} = c_{m1}\,(e^{-k\tau} - e^{-2k\tau}) \text{ , } m_{t,2} = c_{m2}\,(e^{-k\tau} - e^{-4k\tau})$$

  - 3 date-specific functions

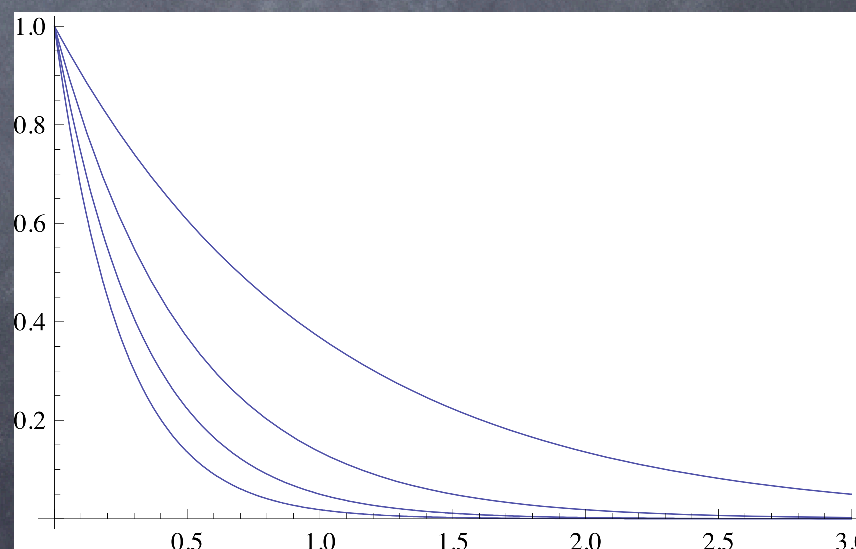
Component Functions

- Basis elements for expressing a model for the yield curve using component SDEs
- Constant k determines shapes

$M_t(\tau)$

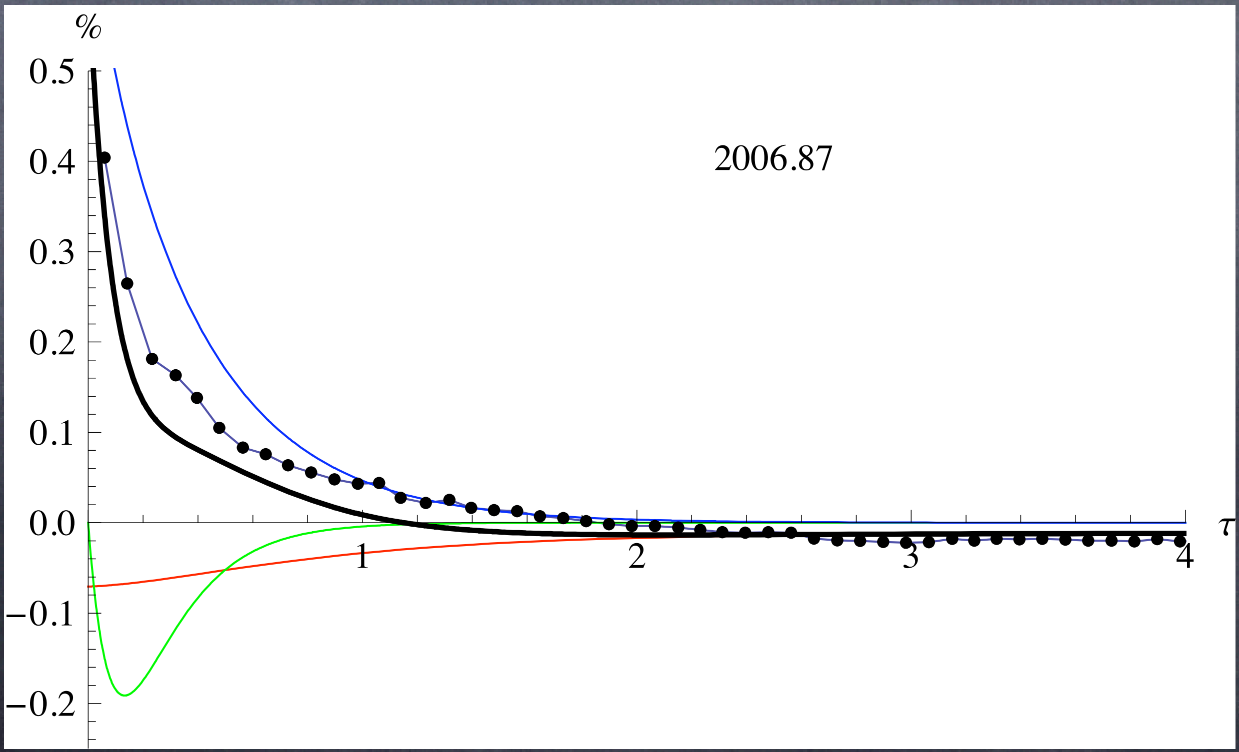


$D_t(\tau)$



Fitted Yield Curves

- Extracted state coefficients for several days
  - Estimates are smoother than those for each day
  - Unconditional, maturity, date components
  - Observation error ought to be uncorrelated



Forecasting

- Extrapolate fit
  - Recursively update as forecast extends beyond initial training period
  - Negative values indicate our models dominate random walk (Newey-West statistic)

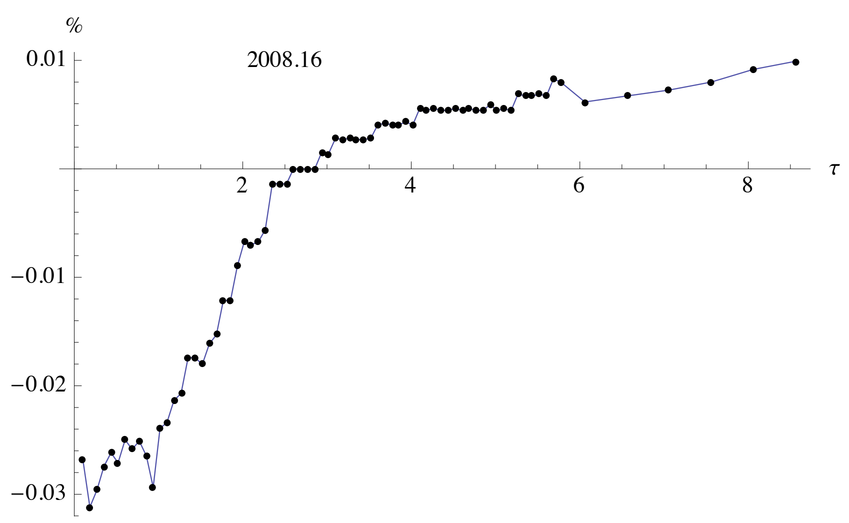
	65 days	125 days	250 days
Crude	2.1	-0.09	-2.2
Heating Oil	-5.8	-8.7	-2.2
Natural Gas	-1.6	-2.2	-1.4
Soybean	-5.1	-7.4	-3.1

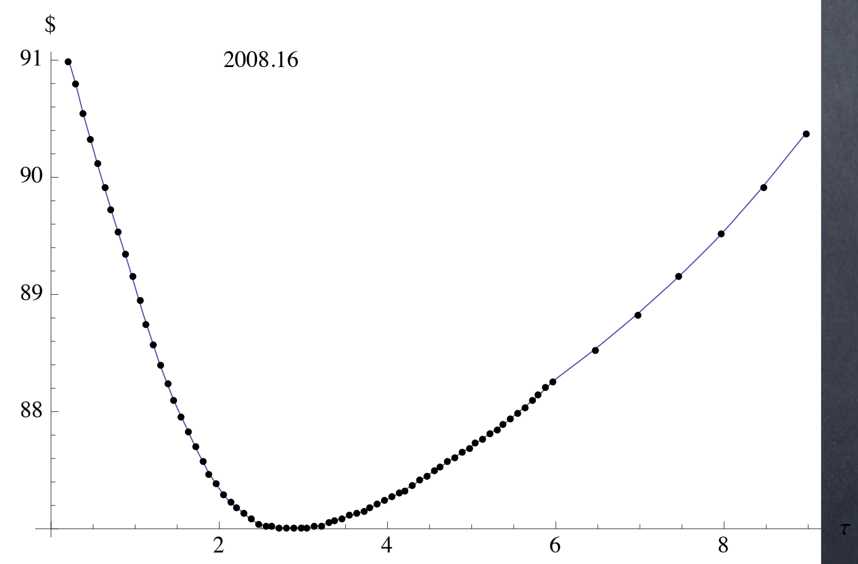
Data

<u>Never</u> underestimate the time that it takes to prepare the data.

Prices

- Yields are not directly observed
  - Obtain prices of futures contracts from data vendor CRB Trader
  - Prices are much smoother





From Price to Yield

- Compute the yield at the midpoint between two observed prices as the continuously compounded rate of interest

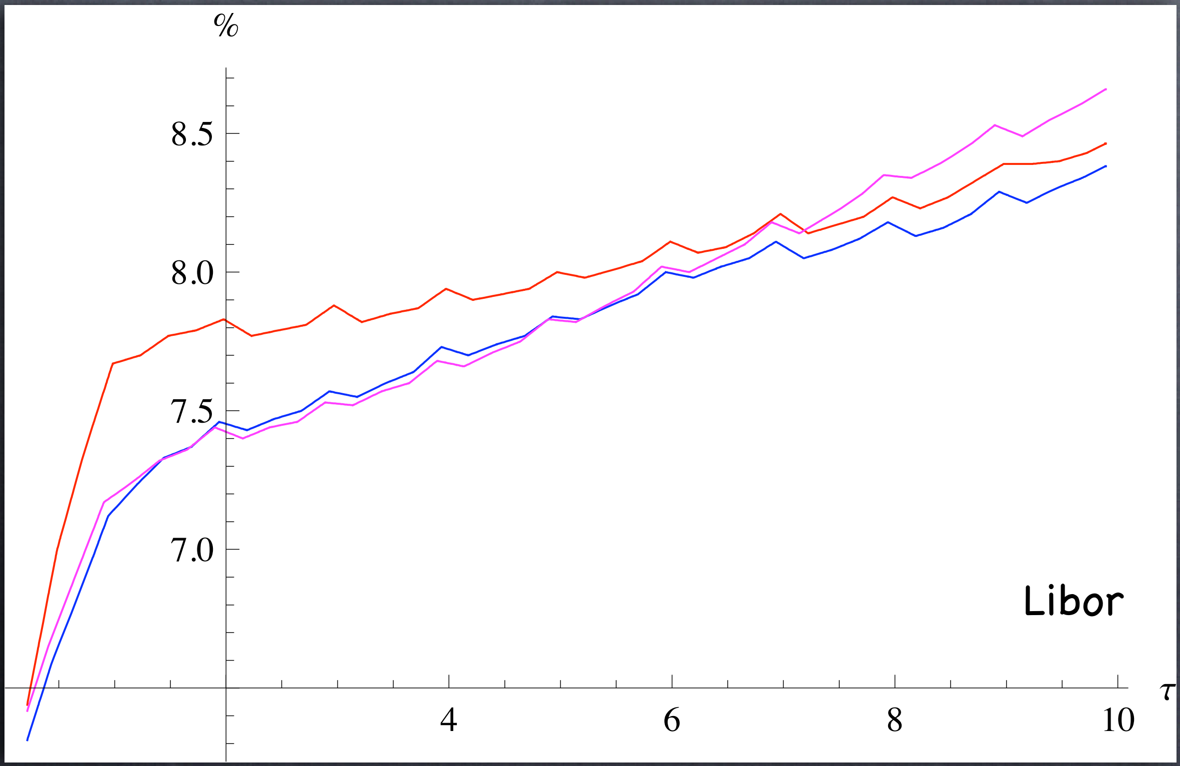
$$y_t(\tau) = \frac{\log p_t(\tau+d) - \log p_t(\tau)}{d} \quad \text{as } d \to 0$$

- Results
  - Differencing magnifies random noise
  - Observed only at discrete set of points
  - Terminal date is about 15 days
  - No observed spot rate
  - Anomalies (aka, "market microstructure")

Anomaly in Cash Yields

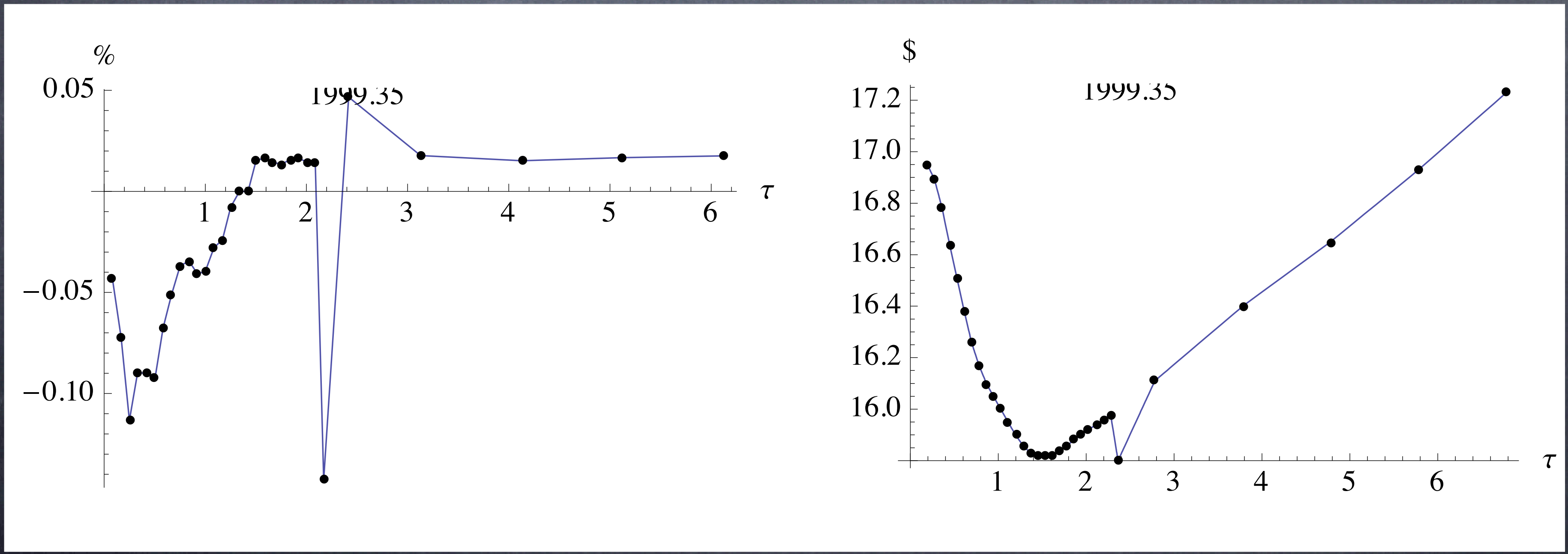
- Yield curves are about 10 days apart
- Ripple induced by preference for contract at a specific date (market microstructure)

1995.28



Anomaly: Sticky Price

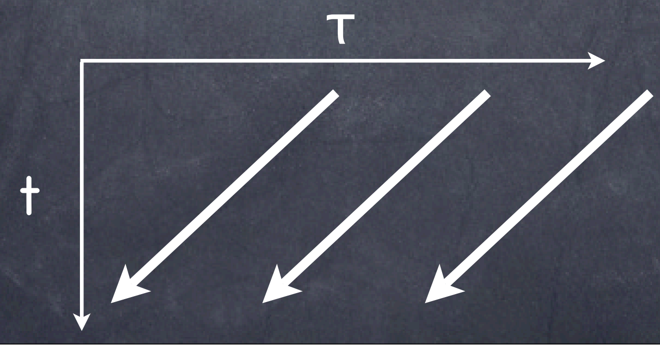
- A contract for light crude did not trade this day, so price stayed same as on prior day
- Rest of the curve shifted



Patching Anomalies

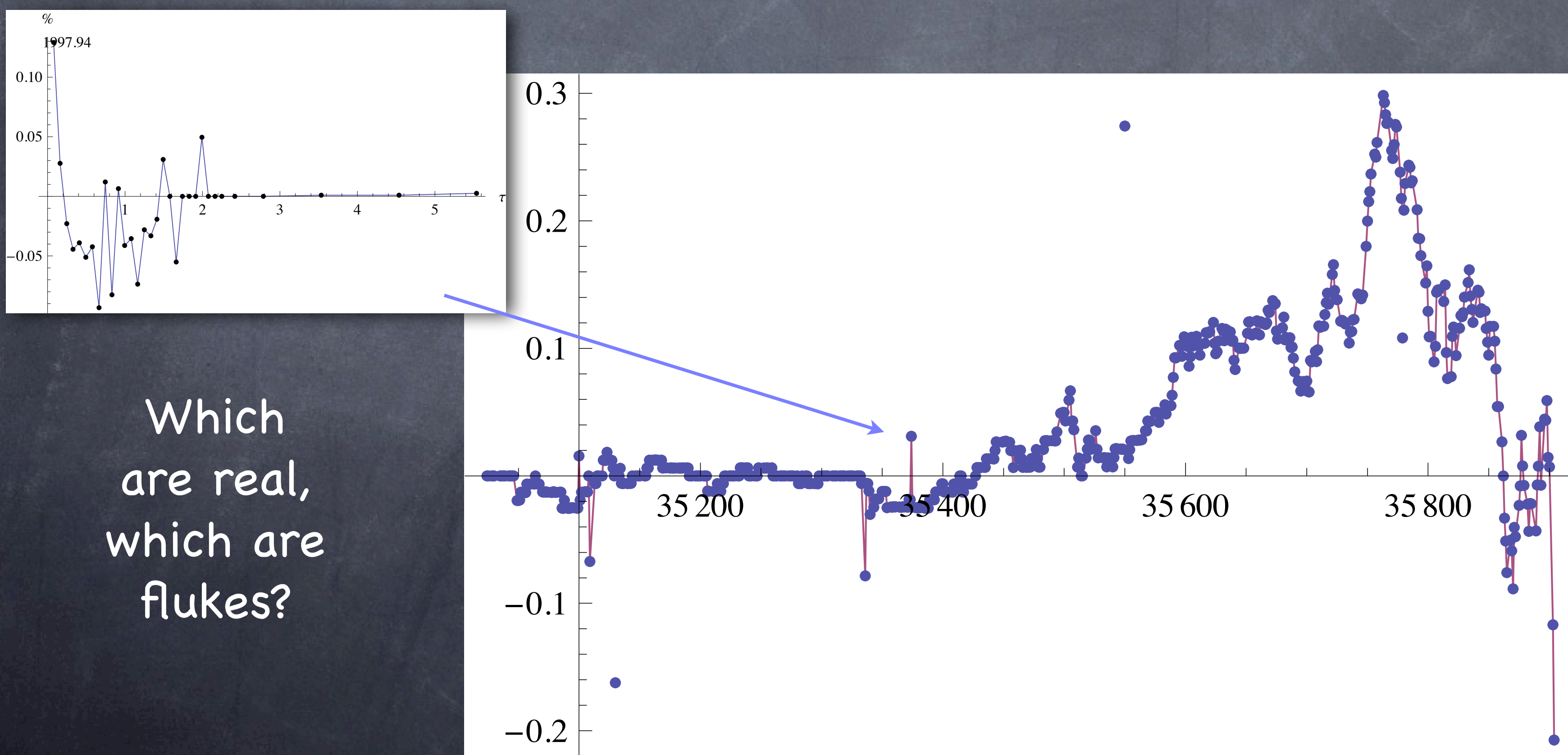
- Adaptive procedure that will not introduce side effects
- Approach is to follow a <u>contract</u> over time rather than fixed maturity
  - Avoid interpolation-induced transitions
  - Contracts have a more consistent sequential pattern than in yields over varying maturities
- Contracts follow a diagonal in the data

  $y_t(\tau)$ for fixed expiry date $t+\tau$



Fixing the Anomalies

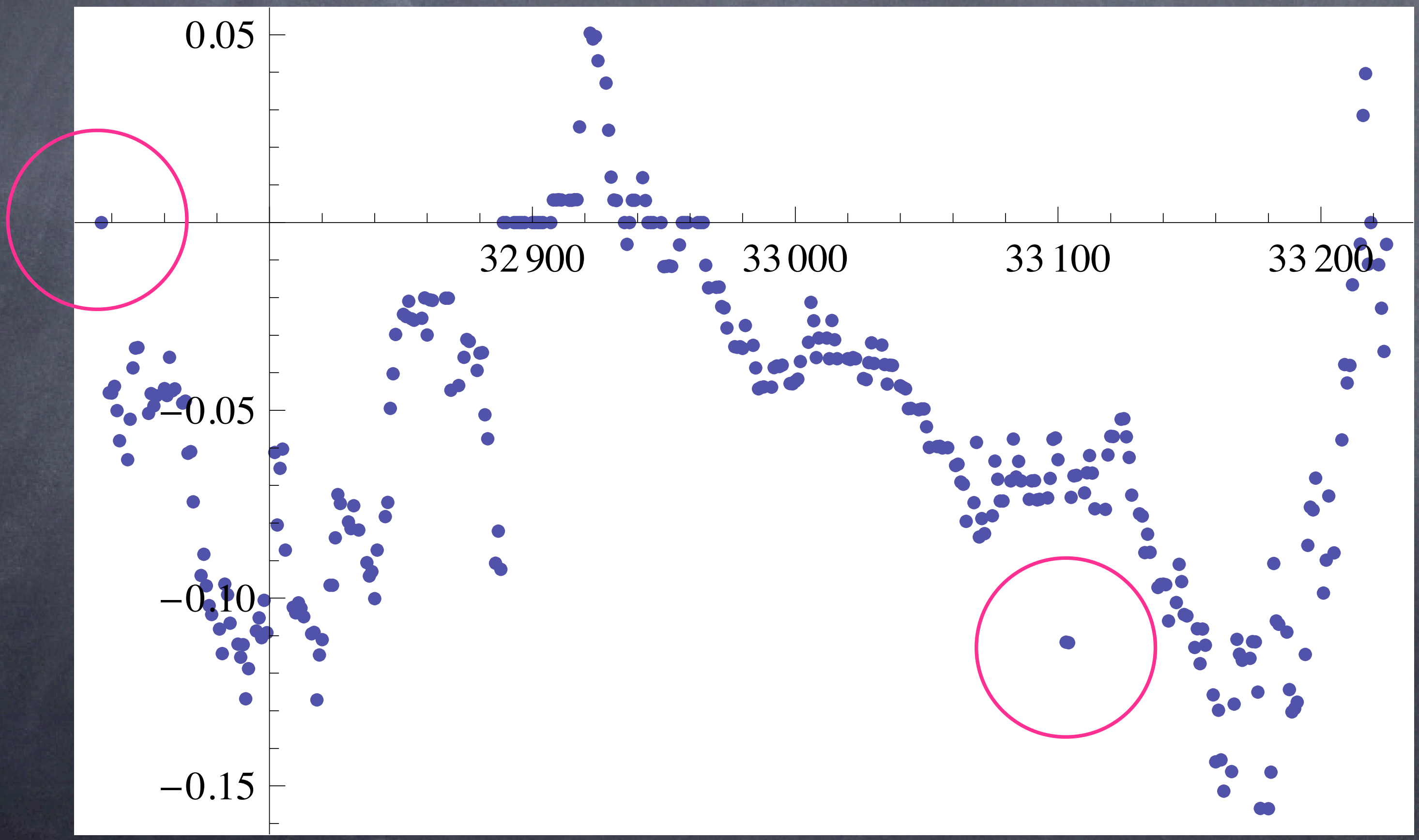
- Follow a contract over time
  - Outliers are more easily identified in the sequence of yields associated with contract rather than over the yield curve itself.



Which are real, which are flukes?

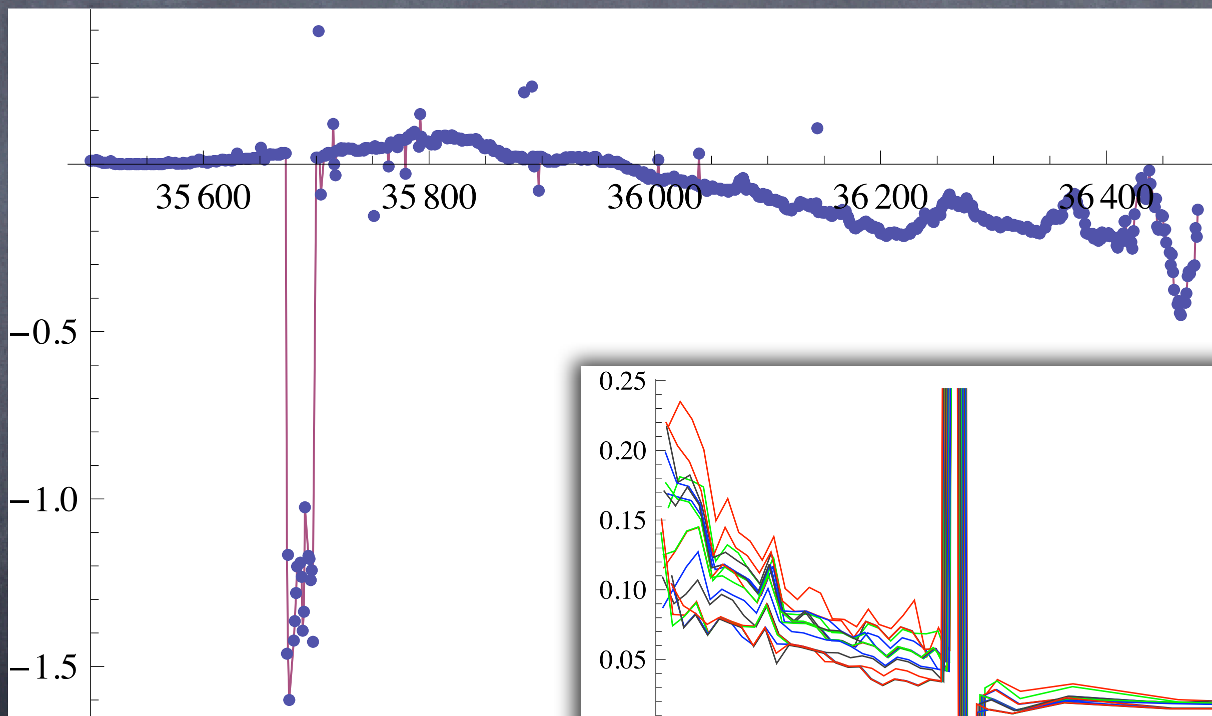
Does not remove all...

- In order not to distort "real" prices, miss a few outliers.

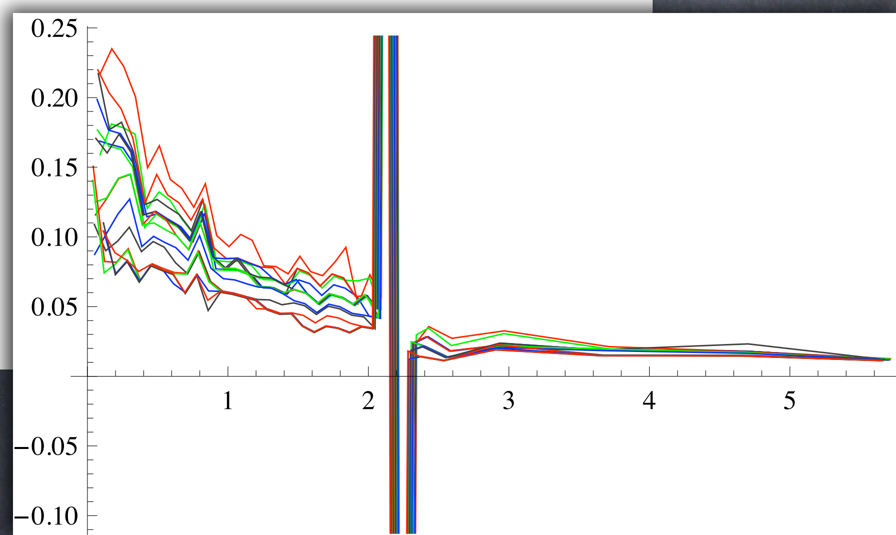


Nor a sustained problem…

- Moving medians of length 3 cannot patch a period with a sustained anomaly.



persists for about 3 weeks



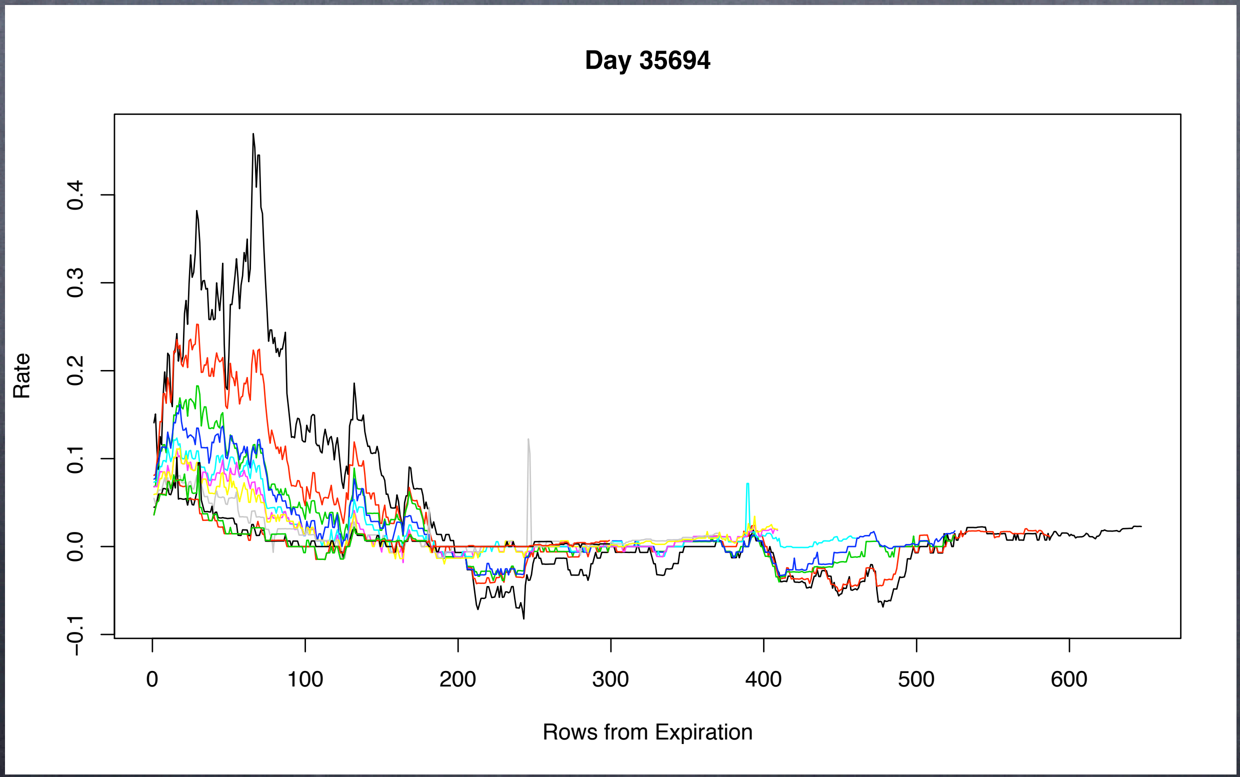
Contracts

Time Series Models

- Structural model
  - Describes the yield curve as a function $f(\tau)$
  - Can evaluate $f(\tau)$ at any maturity
  - Theoretical properties, derived quantities
- Time series
  - Considerable methodology available
  - Cannot hold maturity τ fixed unless were able to observe $y_t(\tau)$ for all t
    - Interpolation between points introduces artifacts
  - Following a single contract provides most direct series

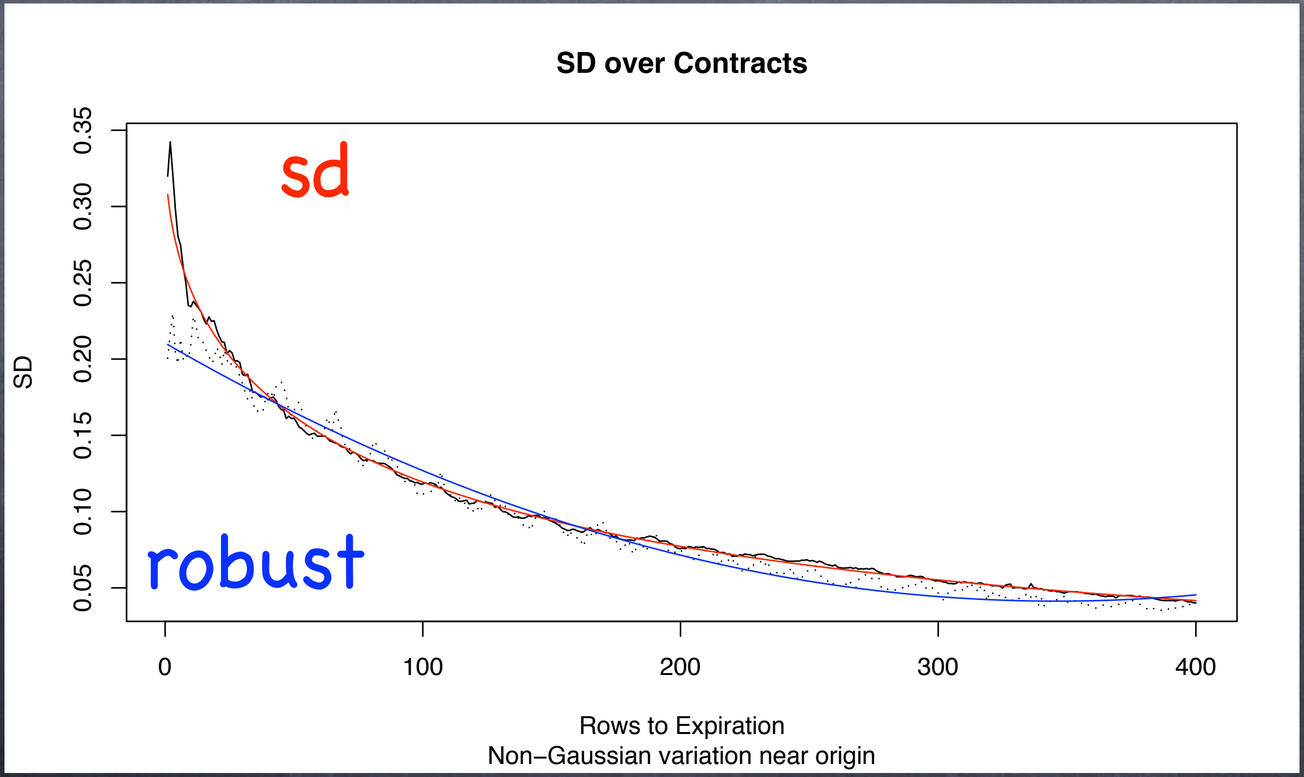
Yields of Contracts

- Think of data as collection of contracts $y_t(\tau)$ identified by expiry date
- Contracts end, becoming more variable as τ approaches 0 where yield is more volatile.



SD of Contracts

- Standard deviation of contracts as τ approaches zero.
  - Very strong presence of outliers as τ nears 0



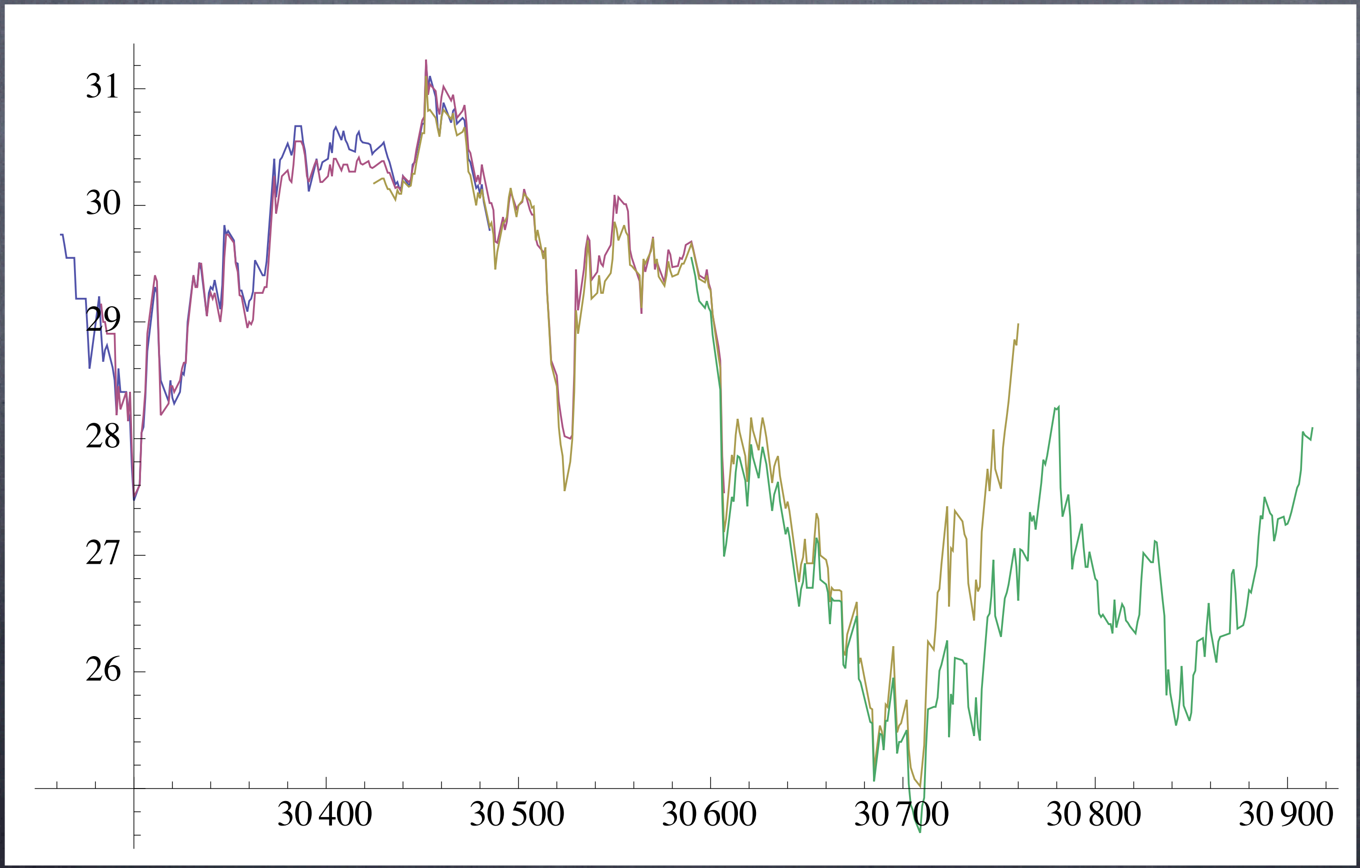
Cointegration

- Differences in yields of adjacent contracts are non-stationary as time gap increases
- Suggests that contracts are cointegrated when not "too far" apart
  - Degree of time separation also measures the rate of change in the yield curve, a sort of stochastic modulus of continuity of the yield
  - Interpret the source of non-stationarity as due to movements of some underlying yield process
  - Latent variable type of model

Prices of Contracts

- Prices again seem simpler to use
- Plot shows prices of 4 contracts for light crude, about 100 days apart

mid 1984



Prices of Contracts

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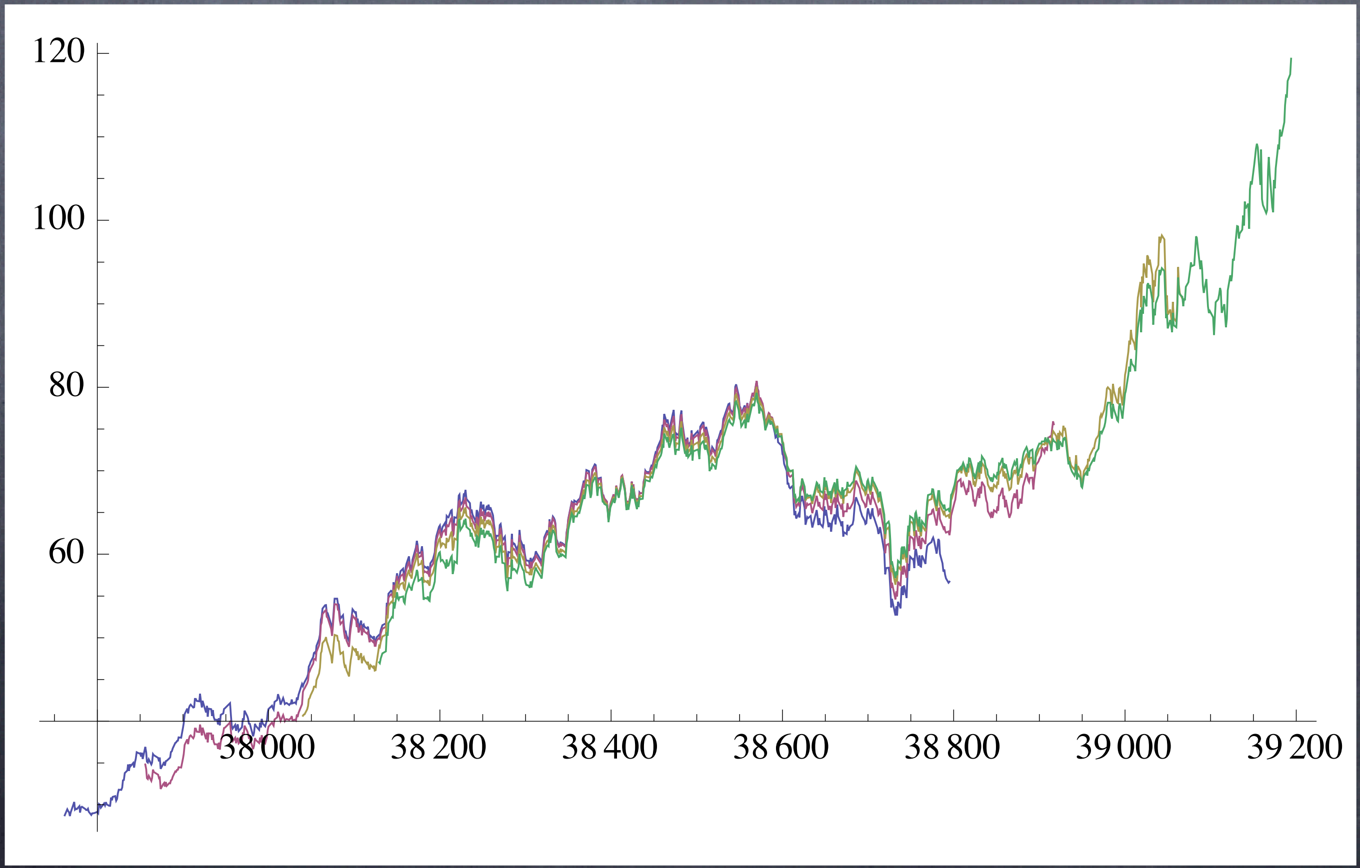
early 2007



More things

- Multivariate structure
  - How to use the evident contemporaneous movements in prices or yields for various products (such as the various soy commodities)
  - Related to movements in yield for cash as well
- Model after subtracting out the yield for cash to extract a convenience yield
  - Implications for stationarity after remove the yield: simpler model?

Summary

- Models for yield curve
  - Capture dynamics of yields
  - Date and maturity specific decomposition relevant for commodities as well
  - Out-of-sample performance superior to random walk benchmark
- Data analysis suggests reasons to work with contracts rather than a function to maturity
  - Outliers
  - More amenable to statistical methods
- Many unresolved questions