Statistics 430
Assignment

Due April 3

This homework assignment considers another use of the central limit theorem. First, it will lead you through the derivation, one that is quite similar to the one done in class. Second, you'll use the resulting normal approximation.

Throughout this assignment, the random variable X denotes a Poisson random variable with parameter λ and PMF given by

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}\ , \quad x = 0, 1, \ldots$$

We'll consider what happens to this density as $\lambda \to \infty$. To keep the algebra manageable, assume that λ is an integer.

1. Show (for integer valued $\lambda > 0$) that the peak (mode) of the Poisson density is at $\lambda - 1$ and at λ. That is, show that $p(\lambda - 1) = p(\lambda)$ and that the common value is the maximum of $p(x)$. (Hint: Look at the ratio of $p(x)/p(x-1)$ for x near λ.)

2. Show that we can write the ratio of probability at $\lambda + x$ (for $x > 0$) to the probability at the mode (which is also the mean) λ as the following sum:
$$\frac{p(\lambda + x)}{p(\lambda)} = \frac{p(\lambda + x)}{p(\lambda + x - 1)}\,\frac{p(\lambda + x - 1)}{p(\lambda + x - 2)} \cdots \frac{p(\lambda + 1)}{p(\lambda)}\ . \tag{1}$$

3. Show that a typical ratio in the product (1) has the form
$$\frac{p(j)}{p(j-1)} = \frac{\lambda}{j}\ .$$

4. Combine the results of the previous two steps to show that the log of the big product in (1) can be written as the sum
$$\log \frac{p(\lambda + x)}{p(\lambda)} = -\sum_{j=1}^{x} \log(1 + j/\lambda)\ . \tag{2}$$

5. Approximate the sum in (2) using $\log(1+\epsilon) \approx \epsilon$ for ϵ "close" to zero. Also approximate the sum of the first x integers as $\sum_{j=1}^{x} j \approx x^2/2$, arriving at
$$\log \frac{p(\lambda + x)}{p(\lambda)} \approx -\frac{x^2}{2\,\lambda}\ . \tag{3}$$

6. Because of the use of the approximation to the log in the prior step, under what conditions (*i.e.*, for what values of x) will the approximation in equation (3) be useful?

7. A very famous approximation for the factorial finishes up the derivation. The approximation is known as Stirling's formula, given by (for positive integers)
$$j! \approx j^j e^{-j} \sqrt{2\pi j}\ . \tag{4}$$

   Compare Stirling's approximation for $j!$ to the exact value for $j = 5, 10, 20$, and 50. How accurate is the approximation?

8. Substitute Stirling's approximation (4) for the factorial in $p(\lambda)$ to show that

$$p(\lambda + x) \approx \frac{e^{-x^2/(2\lambda)}}{\sqrt{2\pi\lambda}} \,. \tag{5}$$

9. (a) Suppose that $X \sim \text{Poisson}(4)$. Compare the normal approximation given by (5) for $P(X = 6)$ to the exact probability given by the Poisson PMF.

   (b) Repeat (a), but with $\lambda = 9$ and $P(X = 12)$.

   (c) Repeat (a), but with $\lambda = 36$ and $P(X = 42)$.

   (d) Repeat (a), but with $\lambda = 100$ and $P(X = 110)$.

10. The bell-shaped normal distribution can be attributed to combining or adding together many small influences. Along these lines, explain why the normal approximation whould "work" in the sense of approximating Poisson probabilities.