Statistics 430

Final Exam (Spring, 2002)

Write your answers in the supplied blue books. You must *show all work* related to the solution in order to receive full credit. The items (not the questions) are approximately equally weighted. Be sure to attempt all items of each question.

1. The Big Lottery is conducted in the following manner. An urn holds 50 balls, numbered from 1 to 50. The balls are mixed, and 6 are drawn in succession, without replacement. In order to win, an entry must match these 6 numbers. Order of the numbers is not important.

   (a) If the holder of any winning ticket is paid \$15 million, then what is the expected value of a "game" in which we purchase a single entry ticket for \$1? If we win, we get \$15 million from the lottery. If we lose, they keep our \$1.

   (b) Suppose that the winning set of numbers is $\{1, 2, 3, 4, 5, 6\}$, and assume that 10,000,000 customers each purchase a single \$1 entry to this lottery by randomly picking their value (independently of each other). Let X denote the number of winning entrants. Find the probability density of X.

   (c) In actuality, the total prize (\$15 million) is split among all winning entrants. Suppose that you have bought a ticket and discover that you are a winner. Assuming 10,000,000 other customers each purchase a single \$1 entry to this lottery (selected randomly and independently), then what is the expected value of your winnings? Use a Poisson approximation to find the answer.

2. Let $X_1, X_2, \ldots, X_{20}$ denote 20 independent random variables that share a common continuous distribution function, F_X. For example, we might produce a set of such values by simulating a random procedure on a computer. Let M denote the median of F_X; that is, $F_X(M) = 0.5$.

   If we denote the ordered values as

   $$X_{(1)} < X_{(2)} < \ldots < X_{(20)} ,$$

   then find $P(X_{(3)} < M < X_{(4)})$. Note that if we define $W_i = 1$ if $X_i < M$ and 0 otherwise, then the event $\{X_{(3)} < M < X_{(4)}\}$ is equivalent to the event $\{\sum_{i=1}^{20} W_i = 3\}$.

3. The diameter D of chocolate-chip cookies produced by an automated bakery is randomly distributed, with a mean of 4 inches and a standard deviation of 1/2 inch. Chips have been pre-mixed into the dough that is mechanically extruded (*i.e.*, squeezed) onto a baking sheet.

   (a) If we assume that the cookies have a uniform thickness of 1/4 inch, then what is the expected amount of batter used to make one cookie (in cubic inches)? (Hint: the volume v of a cylinder of diameter d and height h is $v = h \times \pi(d/2)^2$.)

(b) The number C of chocolate chips in each cookie is random, with about 20 chips (they are small chips) in each cookie, on average. Calculations have shown that the variance of the number of chips per cookie is about 25. The bakery claims "every bag of cookies has at least 1000 chips". If a bag has 48 such cookies, then **estimate** the probability that the claim is true by using the central limit theorem.

(c) Engineers believe that the number of chips is related to the diameter of the cookie. They have estimated the covariance between the diameter and number of chips to be $\text{Cov}(D, C) = -4$. Is this value reasonable? Offer **two** explanations to support your choice.

4. The length of time X in hours required to take an exam is known from past experience to have the following density:

$$f_X(x) = (2e^{-2x/3})/3, \quad x \geq 0.$$

(a) If 3 students take this exam (independently, of course), then what is the probability that the *first* student to complete the exam finishes *after* 1 hour? (*i.e.*, all of them take at least an hour.)

(b) If the first student to complete the exam takes 1 hour, and the last student to complete the exam takes 2 hours, then what is the probability that the second student to finish takes more than 1.5 hours?

5. The annual returns on two speculative investments (like stocks) have the following properties. Both are normally distributed, with annual returns having the following means and variances:

$$\text{Stock A: } \mu_A = 0.10,\ \sigma_A^2 = 0.162 \qquad \text{Stock B: } \mu_B = 0.25,\ \sigma_B^2 = 0.600$$

(a) If an investor purchases 100 shares at \$1 per share of Stock A at the start of the year, then what is the probability that the value of this investment will be *less than* \$30 by the end of the year?

(b) If an investor can invest \$100 in either Stock A or in Stock B and plans to hold the investment for many years, which will be more likely to have higher long-run value?

(c) A portfolio with initial value \$1 was formed by investing \$0.50 in Stock A and \$0.50 in Stock B. If the portfolio has about a 15% chance of being worth more than \$1.70 by the end of the year, then what is the correlation between the returns on these two stocks?

6. Bonus question. Total possible value 5 points.

What portfolio of Stocks A and B as defined in the previous question has maximal long-run value? Assume that the two stocks have correlation $\rho = 0.3$ for this calculation.