Supplement to "A Framework For Estimation of Convex Functions"

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Abstract

In this supplement we prove the additional technical lemmas stated in Section 7.1 which are used in the proofs of the main results.

Lemma 4 *The function H^{-1} defined in Section 2.1 is concave and nondecreasing. It is strictly increasing for all x where $H^{-1}(x) < \frac{1}{2}$. Moreover for $C \geq 1$ it satisfies*

$$H^{-1}(Ct) \leq C^{\frac{2}{3}} H^{-1}(t). \tag{60}$$

The function K defined in Section 2.1 is also increasing and satisfies for $C \geq 1$

$$C^{\frac{2}{3}} K(t) \leq K(Ct) \leq CK(t). \tag{61}$$

Proof of Lemma 4: First note that H is a nondecreasing convex function. Moreover there is a unique point x_0 such that it is strictly increasing on some open interval $(x_0, \frac{1}{2})$ where $f_s(x_0) = 0$. The inverse function $H^{-1}(x)$ is thus strictly increasing on the interval $(0, H(\frac{1}{2}))$. In this interval $H^{-1}(x) < \frac{1}{2}$. For $x > H(\frac{1}{2})$, $H^{-1}(x) = \frac{1}{2}$. It follows that H^{-1} is nondecreasing. The concavity of H^{-1} is guaranteed because it is the inverse of an increasing convex function.

Now let $C \geq 1$. Then since f_s is convex and $f_s(0) = 0$ it follows that whenever $C^{2/3} y \leq \frac{1}{2}$,

$$C^{2/3} f_s(y) \leq f_s(C^{2/3} y)$$

and hence also

$$CH(y) = C\sqrt{y} f_s(y) \leq C^{1/3} \sqrt{y} f_s(C^{2/3} y) = H(C^{2/3} y).$$

*The research of Tony Cai was supported in part by NSF Grant DMS-0604954 and NSF FRG Grant DMS-0854973.

Now let $y = H^{-1}(t)$. Clearly if $C^{2/3}H^{-1}(t) \geq \frac{1}{2}$ then (60) must hold. Hence suppose that $C^{2/3}H^{-1}(t) < \frac{1}{2}$. In this case let $y = H^{-1}(t)$ Then

$$CH(H^{-1}(t)) \leq H(C^{2/3}H^{-1}(t))$$

and hence

$$Ct \leq H(C^{2/3}H^{-1}(t)).$$

Consequently,

$$H^{-1}(Ct) \leq H^{-1}(H(C^{2/3}H^{-1}(t))) = C^{2/3}H^{-1}(t)$$

which establishes (60) in this other case.

Note that for $C \geq 1$,

$$K(Ct) = \frac{Ct}{\sqrt{H^{-1}(Ct)}} \geq \frac{Ct}{C^{1/3}\sqrt{H^{-1}(t)}}.$$

The first inequality in equation (61) and the fact that K is increasing immediately follows. On the other hand,

$$K(Ct) = \frac{Ct}{\sqrt{H^{-1}(Ct)}} \leq \frac{Ct}{\sqrt{H^{-1}(t)}} = CK(t),$$

which yields the second inequality in equation (61). ∎

Lemma 5 *Let f be a nonnegative convex function on $[-\frac{1}{2}, \frac{1}{2}]$. For $d > 0$ let t be the supremum over all y with $f_s(y) \leq d$ where f_s defined in Section 2.1 is the symmetrized and centered version of f. Then there is a convex function g with $g(0) - f(0) = d$ and for which*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (g(x) - f(x))^2 dx \leq \frac{9}{4} d^2 t. \tag{62}$$

It follows that for each $0 \leq t \leq \frac{1}{2}$ there is a convex function g with $g(0) - f(0) = f_s(t)$ such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (g(x) - f(x))^2 dx \leq \frac{9}{4} H^2(t) \tag{63}$$

where the function H is defined in Section 2.1 Moreover for any convex h with $h(0) - f(0) = d > 0$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (h(x) - f(x))^2 dx \geq \frac{2}{3} d^2 t. \tag{64}$$

Remark: The constants $\frac{9}{4}$ and $\frac{2}{3}$ in (62) and (64) are sharp.

Proof of Lemma 5: Throughout this proof we shall without loss of generality take $f(0) = 0$. First suppose that $f_s(\frac{1}{2}) < d$. Then $t = \frac{1}{2}$. In this case take $g(x) = d$ and it is clear that

$$\int_{-1/2}^{\frac{1}{2}} (g(x) - f(x))^2 dx \leq 2d^2$$

and in this case (62) holds.

We must now consider the situation where $f_s(t) = d$ and hence $f(t) + f(-t) = 2d$. We shall consider two cases. In the first $\max(f(t), f(-t)) \geq \frac{3d}{2}$ and in the second case $d \leq \max(f(t), f(-t)) < \frac{3d}{2}$. In the first case for the moment assume that $f(t) \geq \frac{3d}{2}$. Then take $g(x) = \max(f(x), d + \frac{d}{2t}x)$. Note that g is convex as it is a maximum of two convex functions. Also $g(x) = f(x)$ at least for $x \leq -2t$ and $x \geq t$. Moreover since f is nonnegative It is also clear that $g(x) - f(x) \leq d + \frac{d}{2t}x$. Hence in this case

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (g(x) - f(x))^2 dx \leq \int_{-2t}^{t} (d + \frac{d}{2t}x)^2 dx = \frac{9}{4}d^2 t.$$

Similarly when $f(-t) \geq \frac{3d}{2}$ an entirely similar argument can be applied to the function $g(x) = \max(f(x), d - \frac{d}{2t}x)$. Equation (62) of the lemma thus holds under the first case.

In the second case we have $\max(f(t), f(-t)) \leq \frac{3d}{2}$. In this case take $g(x) = \max(f(x), d + \frac{f(t)-f(-t)}{2t}x)$. In this case $g(x) = f(x)$ for $|x| \geq t$ and otherwise $g(x) - f(x) \leq d + \frac{f(t)-f(-t)}{2t}x$. It follows that

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} (g(x) - f(x))^2 dx &\leq \int_{-t}^{t} (d + \frac{f(t) - f(-t)}{2t}x)^2 dx \\
&= \frac{2t}{3}\left(3d^2 + \frac{(f(t) - f(-t))^2}{4}\right) \\
&\leq \frac{2t}{3}\left(3d^2 + \frac{d^2}{4}\right) = \frac{13}{6}d^2 t.
\end{aligned}$$

Thus equation (62) of the lemma also holds in the second case since $\frac{13}{6} \leq \frac{9}{4}$. Equation (63) follows immediately on taking $d = f(t)$ and noting that $t^2 f(t) = H(t)$. We now turn to the proof of (64). For any pair of convex functions f and h let $\tilde{f}(x) = \frac{f(x)+f(-x)}{2}$ and $\tilde{h}(x) = \frac{h(x)+h(-x)}{2}$ be symmetrized versions. Note that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (h(x) - f(x))^2 dx \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} (\tilde{h}(x) - \tilde{f}(x))^2 dx.$$

Note that since $\tilde{h}$ is convex and symmetric with $\tilde{h}(0) = d$ it follows that $\tilde{h}(x) \geq d$ for all $x \in [-1/2, 1/2]$. Hence $\tilde{h}(x) \geq d$. Note also that $f_s(t) \leq d$ and hence for $|x| \leq t$ it follows that $\tilde{f}(x) \leq \frac{|x|}{t}d$ and hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (\tilde{h}(x) - \tilde{f}(x))^2 dx \geq \int_{-t}^{t} (d - \frac{|x|}{t}d)^2 dx = \frac{2}{3}d^2 t$$

and (64) also follows. ∎

Lemma 6 *Let f and g be convex functions with $f(0)-g(0)=a>0$. Let t be the supremum of all y for which $f_s(y)\le a$. Then*

$$\int_{-1/2}^{1/2}(f(x)-g(x))^2dx \ge 0.3ta^2. \tag{65}$$

The proof of this lemma requires an additional technical result which will be stated and proved in Lemma 11 at the end of this supplement.

Proof of Lemma 6: Note that, since as in the proof of lemma 5,

$$\int_{-1/2}^{1/2}(\tilde f(x)-\tilde g(x))^2dx \le \int_{-1/2}^{1/2}(f(x)-g(x))^2dx$$

where $\tilde f(x)=\frac{f(x)+f(-x)}{2}$ and $\tilde g(x)=\frac{g(x)+g(-x)}{2}$, it suffices to prove the lemma for all symmetric convex functions f and g. Hence we shall assume f and g to be convex, even functions and without loss of generality we shall also take $f(0)=0$ and hence $g(0)=-a$ and $f(t)\le a$. First suppose that $g(x)\le 0$ for $0\le x\le \frac{t}{2}$. Then for $0\le x\le\frac{t}{2}$, $g(x)\le\frac{2a}{t}(x-\frac{t}{2})$. In this case

$$\int_{-1/2}^{1/2}(f(x)-g(x))^2dx \ge \int_{-t/2}^{t/2}\frac{4a^2}{t^2}(x-\frac{t}{2})^2dx = \frac{1}{3}ta^2$$

and the Lemma would hold in this case. So suppose that $g(t_1)=0$ where $t_1<\frac{t}{2}$. In this case f and g must meet in one and only one point. Suppose that $f(t_2)=g(t_2)$. Now let $h(x)=-a+\frac{f(t_2)+a}{t_2}x$. Note that $g(x)\le h(x)\le f(x)$ for $0\le x\le t_2$ and that $g(x)\ge h(x)\ge f(x)$ for $t_2\le x\le\frac{1}{2}$. It follows that

$$\int_{-1/2}^{1/2}(f(x)-g(x))^2dx = 2\int_0^{1/2}(f(x)-g(x))^2dx \ge 2\int_0^t(f(x)-h(x))^2dx.$$

Now let

$$k(x)=\max\left\{\frac{f(t)-f(t_2)}{t-t_2}(x-t)+f(t),\ 0\right\}.$$

Since $f(x)\ge k(x)\ge h(x)$ for $0\le x\le t_2$ and $h(x)\ge k(x)\ge f(x)$ for $t_2\le t$ it follows that

$$\int_{-1/2}^{1/2}(f(x)-g(x))^2dx \ge 2\int_0^t(k(x)-h(x))^2dx$$

Now let $y=\frac{x}{t}$. Note that $k(0)=0$ and $h(0)=-a$ and that k and h are of the form of the functions in Lemma 11. It then follows from this lemma that

$$2\int_0^t(k(x)-h(x))^2dx = 2\int_0^1(k(ty)-h(ty))^2tdy \ge 2*0.1572ta^2 \ge 0.3ta^2 \quad ∎$$

Lemma 7 *Set* $\sigma_{j_*} = \frac{2^{(j_*-1)/2}}{\sqrt{n}}$. *Let* j_* *be defined as in Section 4, then*

$$ET_{j_*} \le \min(E\delta_{j_*} - f(0), \sigma_{j_*}). \tag{66}$$

For $k \ge 1$,

$$E\delta_{j_*-k} - f(0) \ge 2^{k-\frac{3}{2}}\sigma_{j_*} \tag{67}$$

and

$$ET_{j_*-k} \ge \frac{2^{k-1}}{\sqrt{6}}\sigma_{j_*}. \tag{68}$$

Proof of Lemma 7: The proof of this lemma will partly use Lemma 4. First note that Lemma 4 gives $ET_{j_*} \le E\delta_{j_*} - f(0)$ and so (66) is clear in the case that $E\delta_{j_*} - f(0) \le \sigma_{j_*}$. In the case $E\delta_{j_*} - f(0) = \lambda\sigma_{j_*}$ where $\lambda > 1$ note that since δ_{j_*} has the smallest mean squared error it follows that

$$E\delta_{j_*+1} - f(0) \ge \sqrt{(\lambda^2 - 1)}\sigma_{j_*}$$

and hence

$$ET_{j_*} = E\delta_{j_*} - E\delta_{j_*+1} \le (\lambda - \sqrt{(\lambda^2 - 1)})\sigma_{j_*}.$$

This last expression is a decreasing function in λ when $\lambda \ge 1$ and so

$$ET_{j_*} \le \sigma_{j_*}$$

showing (66) in this other case. Now suppose that

$$E\delta_{j_*} - f(0) = \lambda\sigma_{j_*}.$$

Then

$$(E\delta_{j_*-1} - f(0))^2 + \frac{2^{j_*-2}}{n} \ge (E\delta_{j_*} - f(0))^2 + \frac{2^{j_*-1}}{n}$$

and hence

$$E\delta_{j_*-1} - f(0) \ge \sqrt{\frac{1}{2} + \lambda^2}\sigma_{j_*}$$

and

$$ET_{j_*-1} \ge (\sqrt{\frac{1}{2} + \lambda^2} - \lambda)\sigma_{j_*}.$$

Then equation (67) immediately follows from Lemma 4. Note also that Lemma 4 also yields $ET_{j_*-1} \ge \lambda\sigma_{j_*}$. Now the function

$$h(\lambda) = (\sqrt{\frac{1}{2} + \lambda^2} - \lambda).$$

is decreasing in λ and so the maximum of $h(\lambda)$ and λ occurs when $h(\lambda) = \lambda$ which has a solution of $\lambda = \frac{1}{\sqrt{6}}$ It then follows that $ET_{j_*-1} \ge \frac{1}{\sqrt{6}}$ and it follows from Lemma 4 that for $k \ge 1$ $ET_{j_*-1} \ge 2^{k-1}\frac{1}{\sqrt{6}}$ yielding equation (68). ∎

Lemma 8 *For $b > 0$ let t_b be the supremum over all t where $f_s(t) \leq br_n^{\frac{1}{2}}(f)$. Then*

$$t_b \leq \frac{2}{4-b^2}\frac{1}{nr_n(f)} \tag{69}$$

and for $b \geq \frac{2}{\sqrt{3}}$,

$$t_b \leq \frac{b3\sqrt{3}}{8nr_n(f)}. \tag{70}$$

Proof of Lemma 8: For $b > 0$ let t_b be the supremum of all points t where $f_s(t) \leq br_n^{\frac{1}{2}}(f)$. Note that

$$\frac{1}{2nt_b} + \frac{b^2 r_n(f)}{4} \geq \frac{1}{2nt_b} + (\frac{1}{t_b}\int_0^{t_b} f_s(t)dt)^2 \geq r_n(f).$$

Hence

$$t_b \leq \frac{2}{4-b^2}\frac{1}{nr_n(f)}$$

establishing (69).

Now for $b \geq \frac{2}{\sqrt{3}}$ the convexity of f_s as well as the fact that $f_s(0) = 0$ also gives the bound $t_b \leq \frac{\sqrt{3}b}{2} t_{\frac{2}{\sqrt{3}}}$ and subsituting the bound from (69) for $t_{\frac{2}{\sqrt{3}}}$ then yields (70). ∎

Lemma 9 *Let $\lambda = \sqrt{2}$ and let $h(x)$ be the function given by*

$$\begin{aligned} h(x) &= P(Z \leq \lambda - \frac{x}{2}) + \frac{0.649}{1+x^2} + \frac{1}{4}\frac{x^2}{1+x^2} \\ &+ \frac{1}{1+x^2}\sum_{m=1}^{\infty}(2^m\sqrt{3} + 2^{-m/2}2x)\left(P(Z \leq \lambda)\prod_{l=0}^{m-1} P(Z > \lambda - 2^{-3l/2}\min(x,1))\right)^{1/2}. \end{aligned}$$

Then

$$\sup_{0 \leq x \leq \frac{2}{\sqrt{3}}} h(x) \leq 4.7.$$

Proof of Lemma 9: Note that $h(x)$ is a univariate continuous function. This bound can be verified through direct numerical calculations. ∎

Lemma 10 *Let $g_m(x,y) = (x^2 + 2^{-m})P(Z \leq \lambda - 2^{m/2}(x-y))$. Then for $m \geq 2$ and $y \geq 2^{m-3/2}$*

$$\sup_{x \geq 2y} g_m(x,y) = (4y^2 + 2^{-m})P(Z \leq \lambda - 2^{m/2}y) \tag{71}$$

$$\sup_{x \geq 2y, y \geq 2^{m-3/2}} g_m(x,y) = (2^{2m-1} + 2^{-m})P(Z \geq 2^{3(m-1)/2} - \lambda). \tag{72}$$

Moreover

$$\sup_{x\geq 2y, y\geq\sqrt{2}} g_2(x,y) \leq 0.649 \tag{73}$$

$$\sup_{x\geq \max(\frac{1}{\sqrt{2}},2y), y\geq 0} \frac{g_1(x,y)}{1+y^2} \leq 1.2. \tag{74}$$

Proof of Lemma 10: For fixed $m \geq 2$ and $y \geq 2^{m-3/2}$, write $g_m(x,y)$ as a function of x,

$$\begin{aligned} g_m(x,y) &= x^2 P(Z \leq (\lambda + 2^{m/2}y) - 2^{m/2}x) + 2^{-m} P(Z \leq (\lambda + 2^{m/2}y) - 2^{m/2}x) \\ &= h_1(x) + h_2(x). \end{aligned}$$

The second term $h_2(x)$ is clearly decreasing in x and hence for $x \geq 2y$,

$$\sup_{x\geq 2y} h_2(x) = 2^{-m} P(Z \leq \lambda - 2^{m/2}y).$$

Now let us consider $h_1(x)$. Set $\tau = 2^{m/2}$ and $\gamma = \lambda + 2^{m/2}y$. Then $h_1(x) = x^2 P(Z > \tau x - \gamma)$. Then

$$h_1'(x) = 2xP(Z > \tau x - \gamma) - \tau x^2 \phi(\tau x - \gamma).$$

Hence $g'(x) \leq 0$ if

$$P(Z > \tau x - \gamma) \leq \frac{\tau x}{2}\phi(\tau x - \gamma).$$

It follows from the fact $P(Z > z) \leq z^{-1}\phi(z)$ for $z > 0$ that $h_1'(x) < 0$ if

$$\tau x(\tau x - \gamma) \geq 2.$$

This holds for

$$x \geq \frac{\gamma\tau + \sqrt{\gamma^2\tau^2 + 8\tau^2}}{2\tau^2} = \frac{\gamma + \sqrt{\gamma^2+8}}{2\tau}. \tag{75}$$

We only need to verify $2y \geq \frac{\gamma+\sqrt{\gamma^2+8}}{2\tau}$ or equivalently

$$4\tau y \geq \gamma + \sqrt{\gamma^2+8}. \tag{76}$$

Write $z = 2^{m/2}y$. Then (76) is equivalent to

$$4z \geq (\sqrt{2}+z) + \sqrt{(\sqrt{2}+z)^2 + 8}$$

which is the same as $z^2 - \sqrt{2}z - 1 \geq 0$. This last inequality holds for all $z \geq (\sqrt{2}+\sqrt{6})/2$. Note that $m \geq 2$ and so

$$z = 2^{m/2}y \geq 2^{3m/2-3/2} \geq 2\sqrt{2} \geq (\sqrt{2}+\sqrt{6})/2.$$

This proves (71).

We have shown that $h_1(x) = x^2 P(Z > \tau x - \gamma)$ is decreasing for $x \geq \frac{\gamma + \sqrt{\gamma^2+8}}{2\tau}$. It then follows easily that $(4y^2 + 2^{-m})P(Z \leq \lambda - 2^{m/2}y)$ is decreasing in y for $y \geq 2^{m-3/2}$ and so

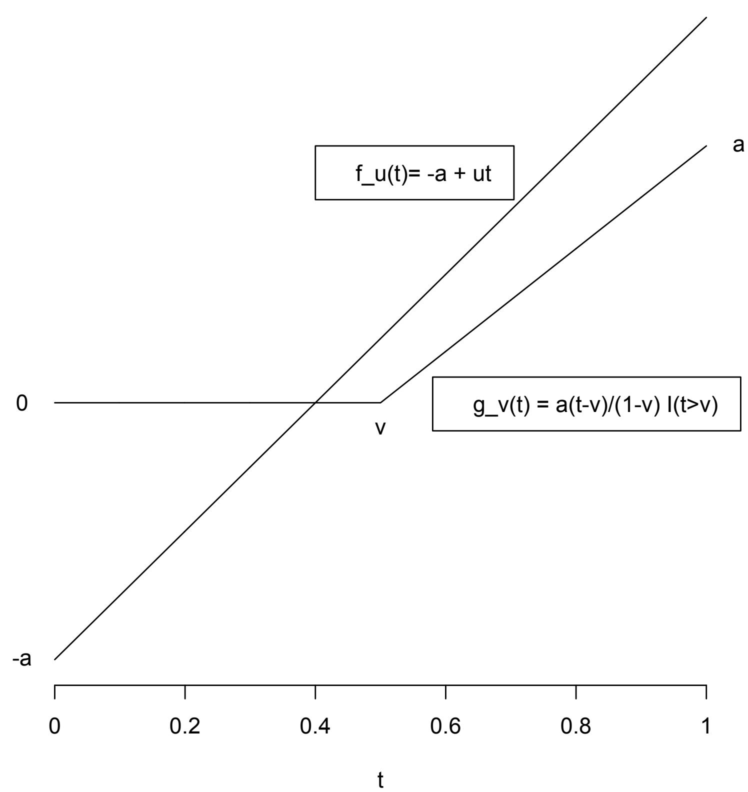
$$\begin{aligned} \sup_{y \geq 2^{m-3/2}} (4y^2 + 2^{-m})P(Z \leq \lambda - 2^{m/2}y) &= (2^{2m-1} + 2^{-m})P(Z \geq 2^{3(m-1)/2} - \lambda) \\ &\leq \frac{2^{2m-1} + 2^{-m}}{2^{3(m-1)/2} - \lambda}\phi(2^{3(m-1)/2} - \lambda), \end{aligned}$$

where $\phi(\cdot)$ is the density function of the standard normal distribution. It is also easy to check that $(4y^2 + 2^{-2})P(Z \leq \lambda - 2y)$ is also decreasing in y and so

$$\sup_{y \geq \sqrt{2}} (4y^2 + 2^{-2})P(Z \leq \lambda - 2y) = \frac{33}{4}P(Z \geq \sqrt{2}) < 0.649.$$

For $m = 1$, (74) can be verified through direct numerical calculations. ∎

Finally, we state and prove the following result which was used in the proof of Lemma 6. This lemma is useful in obtaining a lower bound for the local modulus of continuity. It is helpful to first plot the functions involved.



Lemma 11 *Fix $a > 0$. Let $u > 0$ and $0 \le v < 1$. Define $f_u(t) = -a + ut$ and $g_v(t) = \frac{a}{1-v}(t - v) \cdot I(t \ge v)$. Then*

$$\inf_{u \ge 0, 0 \le v < 1} \int_0^1 (f_u(t) - g_v(t))^2 dt \ge 0.1572a^2. \tag{77}$$

Proof of Lemma 11: Set $S = \int_0^1 (f_u(t) - g_v(t))^2 dt$. Then

$$\begin{aligned} S &= \int_0^v (-a+ut)^2 dt + \int_v^1 (-a + ut + \frac{av}{1-v} - \frac{at}{1-v})^2 dt \\ &= \int_0^v (a^2 - 2aut + u^2t^2) dt \\ &\quad + \frac{1}{(1-v)^2} \int_v^1 \left((u - uv - a)^2 t^2 + 2a(2v-1)(u-uv-a)t + a^2(2v-1)^2\right) dt \\ &= a^2 v - auv^2 + \frac{1}{3}u^2v^3 + \frac{1}{1-v} \cdot \Delta \end{aligned}$$

where

$$\Delta = \frac{1}{3}(u - uv - a)^2(1 + v + v^2) + a(2v-1)(u - uv - a)(1 + v) + a^2(2v-1)^2.$$

We shall first simplify Δ. Setting $w = 1 - v$. Tedious but straightforward algebra shows that

$$\Delta = w\left\{\frac{1}{3}(u^2w - 2au)(w^2 - 3w + 3) + \frac{7}{3}a^2 w + au(2w^2 - 5w + 2)\right\}.$$

Combining this with other terms yields

$$\begin{aligned} S &= a^2(1-w) - au(1-w)^2 + \frac{1}{3}u^2(1-w)^3 \\ &\quad + \frac{1}{3}(u^2w^3 - (2au + 3u^2)w^2 + (6au + 3u^2)w - 6au) \\ &\quad + 2auw^2 + (\frac{7}{3}a^2 - 5au)w + 2au \\ &= \frac{1}{3}auw^2 + (\frac{4}{3}a^2 - au)w + (a^2 - au + \frac{1}{3}u^2). \end{aligned}$$

Note that a is fixed and so S is a function of u and w. We wish to minimize $S = S(u, w)$ with respect to u and w. Setting the partial derivatives to 0, we have

$$\left\{ \begin{array}{rcl} \frac{2}{3}auw + \frac{4}{3}a^2 - au &=& 0 \\ \frac{1}{3}aw^2 - aw - a + \frac{2}{3}u &=& 0 \end{array} \right. .$$

This yields

$$\left\{ \begin{array}{c} u = (-\frac{1}{2}w^2 + \frac{3}{2}w + \frac{3}{2})a \\ 2w^3 - 9w^2 + 3w + 1 = 0 \end{array} \right. .$$

The cubic equation has a unique root between 0 and 1, $w = 0.5986$ and the corresponding value of u is $u = 2.2187a$. The minimum value of S is $S = S(2.2187, 0.5986) = 0.1572a^2$. ∎