OPTIMAL DIFFERENTIALLY PRIVATE PCA AND ESTIMATION FOR SPIKED COVARIANCE MATRICES

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Estimating a covariance matrix and its associated principal components is a fundamental problem in contemporary statistics. While optimal estimation procedures have been developed with well-understood properties, the increasing demand for privacy preservation introduces new complexities to this classical problem. In this paper, we study optimal differentially private Principal Component Analysis (PCA) and covariance estimation within the spiked covariance model.

We precisely characterize the sensitivity of eigenvalues and eigenvectors under this model and establish the minimax rates of convergence for estimating both the principal components and covariance matrix. These rates hold up to logarithmic factors and encompass general Schatten norms, including spectral norm, Frobenius norm, and nuclear norm as special cases.

We introduce computationally efficient differentially private estimators and prove their minimax optimality, up to logarithmic factors. Additionally, matching minimax lower bounds are established. Notably, in comparison with existing literature, our results accommodate a diverging rank, necessitate no eigengap condition between distinct principal components, and remain valid even if the sample size is much smaller than the dimension.

1. Introduction. The covariance structure plays a fundamental role in multivariate analysis, and Principal Component Analysis (PCA) is a widely recognized technique known for its efficacy in dimension reduction and feature extraction [4]. PCA is particularly adept in settings where the data is high-dimensional but the underlying signal displays a low-dimensional structure. The estimation of covariance matrices and principal components finds applications across a diverse spectrum, encompassing tasks such as image recognition, data compression, clustering, risk management, portfolio allocation, mean tests, independence tests, and correlation analysis. Methodologies and theoretical advancements, including minimax optimality, for covariance matrix estimation and PCA, have been well-established in both low-dimensional and high-dimensional settings. See, for example, [6, 9, 10, 15, 26, 30, 39, 41, 44, 51]. For a survey on optimal estimation of high-dimensional covariance structures, see [11].

Amidst the increasing availability of large datasets containing sensitive personal information, privacy concerns in statistical data analysis have gained heightened prominence. The utilization of personal information in statistical analyses raises apprehensions about the potential compromise of individual privacy. Consequently, there is a growing emphasis on developing methodologies and techniques that offer robust privacy guarantees while still facilitating accurate statistical insights. This motivates a comprehensive exploration of the optimal tradeoff between privacy and accuracy in fundamental statistical problems, including PCA and covariance matrix estimation.

Differential privacy (DP), a concept introduced by [22], provides a framework for safeguarding individual privacy in statistical analysis. DP has become a commonly accepted standard in both industrial and governmental applications [1, 2, 19, 24, 42]. The goal of the present paper is to develop methods and optimality results for PCA and covariance matrix estimation within the framework of the spiked covariance model under DP constraints.

1.1. *Problem formulation.* We begin by formally introducing the spiked covariance model and general formulation of the privacy constrained estimation problems.

The spiked covariance structure [26, 27] naturally arises from factor models with homoscedastic noise and has found diverse applications in signal processing, chemometrics, econometrics, population genetics, and various other fields. See, for example, [25, 33, 36, 38]. The spiked covariance model assumes that the population covariance matrix can be decomposed as

$$\Sigma = U\Lambda U^\top + \sigma^2 I_p, \tag{1}$$

where $U \in \mathbb{O}_{p,r}$ and $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_r)$ represent the leading eigenvectors and eigenvalues (excluding σ^2), respectively. Here, $\mathbb{O}_{p,r}$ denotes the set of $p \times r$ matrices satisfying $U^\top U = I_r$. The spiked covariance model is convenient for studying the distribution of sample eigenvalues and eigenvectors, which play a critical role in the statistical inference of Σ and its eigenvectors. For instance, [20] studied the optimal shrinkage of sample eigenvalues in the spiked covariance model. In particular, [10] and [51] established the minimax optimal rates

$$\inf_{\widehat{U}} \sup_{\Sigma \in \Theta(\lambda, \sigma^2)} \mathbb{E}\|\widehat{U}\widehat{U}^\top - UU^\top\| \asymp \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\sqrt{\frac{p}{n}};$$

$$\inf_{\widehat{\Sigma}} \sup_{\Sigma \in \Theta(\lambda, \sigma^2)} \mathbb{E}\|\widehat{\Sigma} - \Sigma\| \asymp \lambda\sqrt{\frac{r}{n}} + \sqrt{\sigma^2(\lambda + \sigma^2)}\sqrt{\frac{p}{n}}, \tag{2}$$

where the infimum is taken over all possible estimators using the data set $X = (X_1, \cdots, X_n)$ consisting of n observations sampled independently from the spiked covariance model (1), the parameter set $\Theta(\lambda, \sigma^2)$ is defined in (4) with λ being the magnitudes of eigenvalues, and $\|\cdot\|$ denotes the matrix spectral norm.

The concept of differential privacy was first introduced in [22]. For a given dataset X and any $\varepsilon > 0$ and $\delta \in [0, 1)$, a randomized algorithm A that maps X into $\mathbb{R}^{d_1 \times d_2}$ is called (ε, δ)-differentially private $((\varepsilon, \delta)$-DP) over the dataset X if

$$\mathbb{P}\big(A(X) \in \mathcal{Q}\big) \leq e^\varepsilon \mathbb{P}\big(A(X') \in \mathcal{Q}\big) + \delta,$$

for all measurable subset $\mathcal{Q} \subset \mathbb{R}^{d_1 \times d_2}$ and all neighboring data set X'. In the standard definition, a dataset X' is a neighbor of X if they differ by only one datum, i.e., one observation in X is replaced by some other, possibly arbitrary, datum. In the context of PCA and covariance matrix estimation, as observations in X are independently sampled from a common distribution, a neighboring dataset X' is obtained by replacing one datum in X with an independent copy. This facilitates exploration of the statistical properties of the sample data.

Under the (ε, δ)-DP constraint, our goal is to investigate the cost of privacy in PCA and covariance matrix estimation. This includes designing minimax optimal (ε, δ)-DP estimators of the principal components and covariance matrix and establishing the privacy-constrained minimax lower bounds.

1.2. *Main contribution.* In this paper, we establish the minimax optimal rates for PCA and covariance matrix estimation in the spiked model under DP constraints. These rates, up to logarithmic terms, are given by:

(3)

$$\inf_{\widehat{U}\in\mathcal{U}_{\varepsilon,\delta}} \sup_{\Sigma\in\Theta(\lambda,\sigma^2)} \frac{\mathbb{E}\|\widehat{U}\widehat{U}^\top - UU^\top\|_q}{r^{1/q}} \asymp \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{r}}{n\epsilon}\right)\bigwedge 1;$$

$$\inf_{\widehat{\Sigma}\in\mathcal{M}_{\varepsilon,\delta}} \sup_{\Sigma\in\Theta(\lambda,\sigma^2)} \frac{\mathbb{E}\|\widehat{\Sigma} - \Sigma\|_q}{r^{1/q}} \asymp \left(\lambda\left(\sqrt{\frac{r}{n}} + \frac{r^{3/2}}{n\varepsilon}\right) + \sqrt{\sigma^2(\lambda+\sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{\sqrt{r}p}{n\varepsilon}\right)\right)\bigwedge\lambda,$$

where the infimum is taken over all possible (ε,δ)-DP algorithms denoted by $\mathcal{U}_{\varepsilon,\delta}$ for principal components and $\mathcal{M}_{\varepsilon,\delta}$ for the covariance matrix. The expectation is taken with respect to the randomness of both the data and the differentially private algorithm. These rates hold in Schatten-q norms for all $q \in [1,\infty]$, including spectral norm ($q=\infty$), Frobenius norm ($q=2$), and nuclear norm ($q=1$) as special cases. The rank r can grow with respect to p as long as $r \leq p/2$, and the sample size can be much smaller than p as long as the signal-to-noise ratio (SNR) satisfies $\lambda/\sigma^2 \geq C_1(\sqrt{p/n} + p/n)$. This condition is minimal since no consistent estimation is possible when this condition does not hold. To our knowledge, this represents the first comprehensive presentation of minimax optimal rates for PCA and covariance matrix estimation under DP constraints.

Our contributions are manifold. Methodologically, we introduce (ε,δ)-DP estimators for PCA and covariance matrices that are computationally efficient. Specifically, we employ the Gaussian mechanism for the sample spectral projector in differentially private PCA. Notably, our DP estimator for the covariance matrix incorporates a novel design to handle unknown orthogonal rotations. These estimators are shown to achieve minimax optimality, up to logarithmic factors. Theoretically, we provide a comprehensive understanding of the minimax optimal rates for PCA and covariance estimation under privacy constraints, valid across all Schatten norms. The derivation of minimax lower bounds employs Fano's lemma with a differential privacy constraint and the construction of well-separated spectral projectors based on the packing complexity of Grassmannians [31, 50].

Differentially private PCA and covariance estimation are challenging because it is difficulty to characterize a sharp sensitivity bound for the eigenvectors. Our main technical contribution lies in a precise characterization of the sensitivity of the sample spectral projector $\widehat{U}\widehat{U}^\top$, quantifying its deviation when one datum X_i is replaced by an independent copy X_i'. A key technical tool is an explicit spectral representation formula for $\widehat{U}\widehat{U}^\top$ adapted from [47]. We derive a similar formula specifically for the spiked covariance model, which is of independent interest. Based on this sharp sensitivity analysis, we apply the Gaussian mechanism to achieve the upper bounds in (3), up to logarithmic terms.

1.3. *Related work.* Minimax optimal rates under (ϵ,δ)-DP guarantees have been established for several statistical problems, such as mean estimation, linear regression, pairwise comparisons, matrix completion, factorization, generalized linear models (GLMs), and sparse GLMs [8, 12, 13, 17, 46]. Additionally, optimality results have also been developed under local privacy constraints. For example, [21] established minimax rates for mean estimation, GLMs, and nonparametric density estimation, while [40] developed minimax theory for estimating linear functionals under local privacy. It is worth noting that local privacy is a stronger notion of privacy compared to (ε,δ)-DP, and it may not be compatible with high-dimensional problems [21].

Differentially private PCA algorithms were proposed in [7, 16, 23] based on the perturbation mechanism, treating each datum X_i as a fixed vector and investigating the sensitivity of

sample eigenvectors. However, their deterministic sensitivity analysis disregards the statistical properties of sample data, resulting in suboptimal error rates when X_i's are i.i.d. sampled from a common distribution, such as the spiked covariance model. Recently, [34] introduced an online PCA algorithm with DP, providing a much sharper upper bound for differentially private PCA under the spiked covariance model. The online Oja's algorithm in [34] consumes one datum at a time, allowing for an explicit representation formula in the updated estimate of eigenvectors and enabling a study of their sensitivity. However, their bound is valid only for the rank-one case ($r = 1$) and is minimax optimal only when $\lambda \geq \sigma^2$. The optimality of their algorithm for general rank r or $\lambda \ll \sigma^2$ remains unclear. Moreover, the minimax optimal rates for estimating Σ under privacy constraints are still unknown.

1.4. *Organization of the paper.* The rest of the paper is organized as follows. In Section 2, we introduce the Gaussian mechanism and study the sensitivity of the empirical spectral projector under the spiked covariance model. We present a DP algorithm for estimating the spectral projector and spiked covariance matrix in the same section. The upper bounds for our proposed DP algorithms are proven in Section 3, where an explicit spectral representation formula under the spiked covariance model is also developed. Section 4 establishes a differentially private Fano's lemma and minimax lower bounds. Extensions to the settings with diverging conditioning number and sub-Gaussian distributions are discussed in Section 5. The proofs of the main results and some of the key technical lemmas are presented in Section 6. The proofs of additional results and technical lemmas are given in the Supplementary Materials [14].

2. Methodology: Gaussian Mechanism and Sensitivity. Our differentially private PCA and covariance estimation method relies on a precise characterization of the sensitivity for both eigenvectors and eigenvalues under the spiked covariance model. We first focus on Gaussian PCA for technical convenience, with a broader discussion on general sub-Gaussian PCA provided in Section 5.

For brevity, let $X := (X_1, \cdots, X_n)$ represent the $p \times n$ matrix collecting all i.i.d. observations X_i sampled from a centered normal distribution $\mathcal{N}(0, \Sigma)$. The sensitivity of eigenvectors and eigenvalues denotes their perturbation if an observation X_i is replaced by an independent copy X_i' expressed briefly as $X^{(i)} := (X_1, \cdots, X_{i-1}, X_i', X_{i+1}, \cdots, X_n)$. Here, X and $X^{(i)}$ form a pair of neighboring datasets [22]. Notably, the sensitivity is contingent on the covariance matrix Σ.

Through out this paper, we consider the spiked covariance matrix model where Σ is from the following parameter space

(4)
$$\Theta(p, r, \lambda, \sigma^2) = \left\{ \Sigma = U\Lambda U^\top + \sigma^2 I_p : \right.$$

$$\left. U \in \mathbb{O}_{p,r}, \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_r), c_0\lambda \leq \lambda_r \leq \lambda_1 \leq C_0\lambda \right\},$$

where I_p is the identity matrix and and $\mathbb{O}_{p,r}$ refers to the set of matrices with orthonormal columns, i.e., matrices satisfying $U^\top U = I_r$. Thus, our focus is on spiked covariance matrices with a bounded condition number, a common assumption in existing literature [9, 16, 34]. However, our methodology remains valid, and the theoretical framework can be extended to the case of an unbounded condition number, as discussed in Section 5. For simplicity, we use $\Theta(\lambda, \sigma^2)$ without explicitly stating the dimensions p and rank r. Let $\mathcal{P}$ denote the family of normal distributions $\mathcal{N}(0, \Sigma)$ with the population covariance matrix $\Sigma \in \Theta(\lambda, \sigma^2)$. Without loss of generality, we assume that σ^2 is known.

Formally, the sensitivity and Gaussian mechanism are described as follows without proofs. See, for example, [22] for more details. Here, $\| \cdot \|_F$ stands for the matrix Frobenius norm.

LEMMA 2.1 (sensitivity and Gaussian mechanism).   *Let X be a given data set and X' be any neighboring data set of X, i.e., X and X' differs by at most one observation. The sensitivity of a function f that maps X into $\mathbb{R}^{d_1 \times d_2}$ is defined by*

$$(5) \qquad \Delta_f := \sup_{neighboring(X,X')} \|f(X) - f(X')\|_{\mathrm{F}}.$$

Then, for any $\varepsilon > 0$ and $\delta \in [0,1)$, the randomized algorithm A defined by $A(X) = f(X) + Z$ where Z has i.i.d. $\mathcal{N}\big(0, 2\Delta_f^2 \varepsilon^{-2} \log(1.25/\delta)\big)$ entries is (ε, δ)-DP over the dataset X.

The definition of sensitivity in Lemma 2.1 relies on the pair of neighboring data sets. Here, X is simply the data matrix where each column represents one observation. While X and X' differ only by one observation, the sensitivity can still be unbounded if no restriction is posed on the difference, e.g., by replacing one observation of X by infinite. Since X consists of i.i.d. columns under the spiked covariance model, we assume that a neighboring data set X' is obtained by replacing some column of X by its i.i.d. copy throughout this paper.

2.1. *Differentially private estimation by Gaussian mechanism.*   Our DP-estimators of principal components and covariance matrix are built on Gaussian mechanism. Here, we assume that the rank r and nuisance variance σ^2 are known for simplicity. Let $\widehat{U}$ be the top-r eigenvectors of the sample covariance matrix $\widehat{\Sigma} := n^{-1} \sum_{i=1}^{n} X_i X_i^{\top}$ and denote $\widehat{U}\widehat{U}^{\top}$ the sample spectral projector. By Lemma 2.1, differentially private PCA can be obtained by adding Gaussian noise Z to $\widehat{U}\widehat{U}^{\top}$ provided that the entrywise variance of Z dominates the sensitivity of $\widehat{U}\widehat{U}^{\top}$. While publishing $\widehat{U}\widehat{U}^{\top} + Z$ protects privacy, it is certainly not a preferable estimator of principal components as it generally lacks validity as a spectral projector. We therefore take the eigenvectors of $\widehat{U}\widehat{U}^{\top} + Z$ as the ultimate estimator. This choice maintains differential privacy, as the post-processing of a differentially private algorithm retains differential privacy according to well-established results, as discussed in [22].

Algorithm 1 Differentially private PCA and covariance estimation

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Input: data matrix $X = (X_1, \cdots, X_n) \in \mathbb{R}^{n \times p}$; eigenvectors and eigenvalues sensitivity Δ_1 and $\Delta_2 > 0$; rank r; nuisance variance σ^2; privacy budget $\varepsilon > 0, \delta \in (0,1)$.
Output: (ε, δ)-DP estimate of U and Σ.
Compute the sample covariance matrix and top-r eigenvectors:

$$\widehat{\Sigma} \longleftarrow \frac{1}{n}\sum_{i=1}^{n} X_i X_i^{\top} \quad \text{and} \quad \widehat{U} \longleftarrow \mathrm{SVD}_r(\widehat{\Sigma})$$

Compute $(\varepsilon/2, \delta/2)$-DP PCA by adding artificial Gaussian noise:

$$\widetilde{U} \longleftarrow \mathrm{SVD}_r\Big(\widehat{U}\widehat{U}^{\top} + Z\Big) \quad \text{where} \quad Z_{ij} = Z_{ji} \overset{\text{i.i.d.}}{\sim} \mathcal{N}\Big(0, \frac{8\Delta_1^2}{\varepsilon^2}\log\frac{2.5}{\delta}\Big), \quad \forall 1 \le i \le j \le p$$

Compute $(\varepsilon/2, \delta/2)$-DP estimates of eigenvalues up to rotations:

$$\widetilde{\Lambda} \longleftarrow \widetilde{U}^{\top}\big(\widehat{\Sigma} - \sigma^2 I_p\big)\widetilde{U} + E \quad \text{where} \quad E_{ij} = E_{ji} \overset{\text{i.i.d.}}{\sim} \mathcal{N}\Big(0, \frac{8\Delta_2^2}{\varepsilon^2}\log\frac{2.5}{\delta}\Big), \quad \forall 1 \le i \le j \le r$$

Compute (ε, δ)-DP covariance estimate by :

$$\widetilde{\Sigma} \longleftarrow \widetilde{U}\widetilde{\Lambda}\widetilde{U}^{\top} + \sigma^2 I_p$$

Return: $\widetilde{U}$ and $\widetilde{\Sigma}$

Our proposed differentially private PCA and covariance estimation procedures are given in Algorithm 1. The sensitivities Δ_1 and Δ_2 are determined by Lemma 2.3 and Lemma 2.4

in Section 2.2, respectively. However, $\widetilde{U}$ and $\widehat{U}$ are close up to an orthogonal rotation. As a result, our algorithm chooses to add Gaussian noise to $\widetilde{U}^\top \widehat{\Sigma} \widetilde{U}$ instead of the empirical eigenvalues $\widehat{\Lambda} := (\widehat{\lambda}_1, \cdots, \widehat{\lambda}_r)^\top$. The added noise level depends on the sensitivity of $\widetilde{U}^\top \widehat{\Sigma} \widetilde{U}$, within which $\widetilde{U}$ is already differentially private. It thus suffices to study the upper bound of $\|\widetilde{U}^\top (\widehat{\Sigma} - \widehat{\Sigma}^{(i)})\widetilde{U}\|_\mathrm{F} \leq \|\widehat{\Sigma} - \widehat{\Sigma}^{(i)}\|_\mathrm{F}$, which will be established in Lemma 2.4.

Our approach to differentially privately estimating the main covariance term involves separately privatizing the eigenvectors and eigenvalues. This separation is driven by the observation that the relative sensitivity of eigenvalues is significantly larger than that of eigenvectors. Note that a natural estimator of $U(\Lambda + \sigma^2 I_r)U^\top$ is $\widehat{U}\widehat{U}^\top \widehat{\Sigma} \widehat{U}\widehat{U}^\top$. It is possible to characterize the sensitivity of this estimator by directly studying the bound $\|\widehat{U}\widehat{U}^\top \widehat{\Sigma} \widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\widehat{\Sigma}^{(i)}\widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_\mathrm{F}$. However, the sensitivity of eigenvalues will be the dominating factor and force us to add unnecessarily large noise to a $p \times p$ matrix. This delivers a statistically sub-optimal estimator of the spiked covariance matrix.

The estimated eigenvectors $\widetilde{U}$ is $(\varepsilon/2, \delta/2)$-DP and eigenvalues $\widetilde{\Lambda}$ is $(\varepsilon/2, \delta/2)$-DP with high probability. By the composition property of differentially private algorithm, the estimator $\widetilde{U}\widetilde{\Lambda}\widetilde{U}^\top$ is (ε, δ)-DP. The conclusion is formally stated in the following lemma. Recall that $\tilde{r} = (r\lambda + p\sigma^2)/(\lambda + \sigma^2)$ is the effective rank of Σ. Here, λ is regarded as the signal strength.

LEMMA 2.2. *Let the data matrix $X = (X_1, \cdots, X_n)$ consists of i.i.d. columns sampled from $\mathcal{N}(0, \Sigma)$ with $\Sigma \in \Theta(\lambda, \sigma^2)$, $\varepsilon > 0, \delta \in (0, 1)$, and assume $n \geq C_1(r \log n + \log^2 n), 2r + C_1 \log n \leq p$, and $\lambda/\sigma^2 \geq C_1(p/n + \sqrt{p/n})$ for some large absolute constant $C_1 > 0$. If we choose*

$$\Delta_1 = C_2\left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\frac{\sqrt{p(r + \log n)}}{n} \quad \text{and} \quad \Delta_2 = C_3\frac{\lambda(r + \log n) + \sigma^2 p}{n},$$

for some large absolute constants $C_2, C_3 > 0$, then Algorithm 1 is (ε, δ)-DP with probability at least $1 - e^{-c_1 p} - 4n^{-9} - 10^{-20\tilde{r}}$ for some absolute constant $c_1 > 0$.

Compared with existing literature [16, 34], our algorithm does not truncate the observations so that $\|X_i\|$ is essentially unbounded and, as a result, our algorithm is differentially private with high probability. Note that the probability terms n^{-9} and $10^{-20\tilde{r}}$ in Lemma 2.2 can be replaced by n^{-C_3} and $10^{-C_4\tilde{r}}$ with any absolute constants $C_3, C_4 > 0$. While we believe Algorithm 1 can be easily modified, e.g., by an additional trimming procedure, to ensure (ε, δ)-differentially privacy almost surely, which will inevitably introduce more logarithmic factors into the upper bounds, we spare no further efforts to pursue the goal in this paper.

The sensitivities Δ_1 and Δ_2 play a critical role in guaranteeing the differential privacy of Algorithm 1, which shall be developed in next section. The conditions $r \log n + \log^2 n = O(n)$ and $2r + \log n = O(p)$ are mild. The SNR condition $\lambda/\sigma^2 \geq C_1(p/n + \sqrt{p/n})$ is typical in the existing literature of spiked covariance matrix model. See, e.g., [35, 51] and references therein.

2.2. *Sensitivity analysis.* In this section, we analyze the sensitivities of sample eigenvectors and eigenvalues under the spiked covariance model. The data matrix $X = (X_1, \cdots, X_n) \sim \mathcal{N}(0, \Sigma)^{\otimes n}$ for some $\Sigma \in \Theta(\lambda, \sigma^2)$. Similarly, its neighboring data matrix $X^{(i)} = (X_1, \cdots, X_i', \cdots, X_n) \sim \mathcal{N}(0, \Sigma)^{\otimes n}$. Define the sample covariance matrices by

$$\widehat{\Sigma} := \frac{1}{n}\sum_{i=1}^n X_i X_i^\top \quad \text{and} \quad \widehat{\Sigma}^{(i)} := \frac{1}{n}\left(X_i' X_i'^\top + \sum_{j \neq i} X_j X_j^\top\right),$$

Denote $\widehat{U} \in \mathbb{O}_{p,r}$ and $\widehat{U}^{(i)} \in \mathbb{O}_{p,r}$ the top-r left eigenvectors of $\widehat{\Sigma}$ and $\widehat{\Sigma}^{(i)}$, respectively. The sensitivity of sample eigenvectors characterizes the deviation between $\widehat{U}$ and $\widehat{U}^{(i)}$ caused by replacing the i-th observation by its i.i.d. copy. Since eigenvectors are determined up to an orthogonal rotation (note that we allow the eigengap $|\lambda_i - \lambda_j|$ to be zero), a commonly used metric for measuring the distance between eigenvectors is the projection distance defined by $\|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_{\mathrm{F}}$.

The primary challenge in differentially private PCA lies in characterizing a precise upper bound for $\|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)\top}\widehat{U}^{(i)}\|_{\mathrm{F}}$. In most existing works [7, 16, 23], the data matrix X is assumed to be fixed, and its columns are all bounded, denoted as $\|X_i\| \leq \gamma$, where we slightly abuse the notation by letting $\|\cdot\|$ denote the ℓ_2-norm for vectors and γ is a deterministic value. This immediately implies an upper bound $\|\widehat{\Sigma} - \widehat{\Sigma}^{(i)}\| \leq 2\gamma^2/n$ and the sensitivity of $\widehat{U}\widehat{U}^\top$ is guaranteed by the Davis-Kahan theorem.

However, this approach becomes invalid when observations are unbounded and sub-optimal when observations are randomly sampled from a common distribution. A more recent work [34] aimed to exploit the statistical properties of i.i.d. samples to achieve a sharper bound for differentially private PCA. This work focused on the rank-one case ($r = 1$) and the Oja's algorithm, well-known for online PCA, which iteratively updates the estimation with one additional observation. The online fashion of Oja's algorithm in the rank-one case allows for an explicit representation of the eigenvector estimator, enabling a sharp upper bound of the sensitivity to be derived. Consequently, nearly optimal differentially private PCA for the case $r = 1$ was achieved. However, it remains unclear how this approach can be extended to the rank-r case and what the minimax optimal convergence rates are.

We take a fundamentally different approach by directly focusing on $\|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_{\mathrm{F}}$. This task presents two challenges: the spectral projector $\widehat{U}\widehat{U}^\top$ involves a complicated function of the data matrix X, and a sharp perturbation analysis is required for a set of r empirical eigenvectors. Fortunately, we leverage an explicit spectral representation formula adapted from [47] and successfully establish a precise upper bound for $\|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_{\mathrm{F}}$.

For brevity in notation, we assume $n \geq C_1(r \log n + \log^2 n)$ and $2r + C_1 \log n \leq p$. We define $\tilde{r} := \mathrm{tr}(\Sigma)/|\Sigma| = (r\lambda + p\sigma^2)/(\lambda + \sigma^2)$ as the *effective rank*. It is clear that $r \leq \tilde{r} \leq p$.

LEMMA 2.3. *There exist absolute constants* $c_1, C_1, C_2, C_3 > 0$ *such that if* $\lambda/\sigma^2 \geq C_1(p/n + \sqrt{p/n})$, *then with probability at least* $1 - e^{-c_1 p} - 3n^{-9} - 10^{-20\tilde{r}}$,

$$(6) \qquad \max_{i \in [n]} \|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_{\mathrm{F}} \leq C_2\left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\frac{\sqrt{p(r + \log n)}}{n}.$$

The $\log n$ term in upper bound (6) is due to the maximization over n. Nevertheless, the bound is much smaller than that achieved by the deterministic analysis in [7, 16, 23]. Indeed, a direct application of Davis-Kahan theorem yields an upper bound $O(\|\widehat{\Sigma} - \widehat{\Sigma}^{(i)}\|\sqrt{r}/\lambda)$, which is at least in the order $O((r\lambda + p\sigma^2)\sqrt{r}/(n\lambda))$, with high probability. The significant improvement is due to a sharp spectral characterization showing that the difference $\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}$ is mainly contributed by the term $\|U^\top(X_iX_i^\top - X_i'X_i'^\top)U_\perp\|_{\mathrm{F}}/(n\lambda)$. Here, $U_\perp \in \mathbb{O}_{p,p-r}$ denotes the orthogonal complement of U such that $(U, U_\perp)$ is an orthogonal matrix. The proof of Lemma 2.3 is technically involved and deferred to Section 6.2. It is worth noting that the original spectral representation formula developed in [47] is inapplicable here because Σ is not exactly rank-r. Interestingly, we establish a similar spectral representation formula exclusively for spiked covariance matrix, which may be of independent interest. See Lemma 3.1 in Section 3.1.

The sensitivity of eigenvalues is also necessary for constructing differentially private co-variance estimation. Let $\lambda_k(\widehat{\Sigma})$ and $\lambda_k(\widehat{\Sigma}^{(i)})$ denote the k-th largest eigenvalue of $\widehat{\Sigma}$ and $\widehat{\Sigma}^{(i)}$, respectively. Compared to the eigenvectors, the sensitivity of eigenvalues can be eas-ily characterized by Hoffman-Weilandt's inequality. The proof of Lemma 2.4 is deferred to Section 6.3.

LEMMA 2.4. *There exist absolute constants $c_1, C_1, C_2 > 0$ such that with probability at least $1 - n^{-10}$,*

$$(7) \qquad \sum_{k=1}^{p} \left| \lambda_k(\widehat{\Sigma}) - \lambda_k(\widehat{\Sigma}^{(i)}) \right|^2 \leq C_2 \left(\frac{\lambda(r + \log n) + \sigma^2 p}{n} \right)^2,$$

for all $i \in [n]$.

We can regard $\left(\sum_{k=1}^{p} \left(\lambda_k(\widehat{\Sigma}) - \lambda_k(\widehat{\Sigma}^{(i)}) \right)^2 \right)^{1/2} / \lambda$ and $\|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_{\mathrm{F}} / \sqrt{r}$ as the relative sensitivity of eigenvectors and eigenvalues, respectively. Lemmas 2.3 and 2.4 show that the relative sensitivity of eigenvalues can be considerably larger than that of eigenvectors. This insight implies that, when designing a differentially private optimal estimation proce-dure for the population covariance matrix, it is advisable to privatize the eigenvalues and eigenvectors separately, as elaborated in Algorithm 1.

3. Upper Bounds with Differential Privacy.

3.1. *Spectral representation formula.* Our key technical tool is the following spectral representation formula. Recall that $\widehat{U}$ and U denote the top-r eigenvectors of $\widehat{\Sigma}$ and Σ, re-spectively. Denote the deviation matrix by $\widehat{\Delta} = \widehat{\Sigma} - \Sigma$ so that $\widehat{\Sigma} = \Sigma + \widehat{\Delta}$ is viewed as a perturbation of the "signal" matrix Σ. The spectral representation formula was first intro-duced in [47], which, however, requires the "signal" matrix to be exactly rank-r. This is certainly not the case here since Σ is full-rank. Here, we develop the spectral representation formula exclusively for the perturbation of a spiked covariance matrix.

The spectral representation formula is actually deterministic. Let the symmetric matrix $\Delta \in \mathbb{R}^{p \times p}$ be an arbitrary perturbation. Denote $\widehat{U}$ the top-r eigenvectors of $\Sigma + \Delta$ where $\Sigma = U\Lambda U^\top + \sigma^2 I_p$ with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_r)$. We are interested in developing an explicit representation formula for the spectral projector $\widehat{U}\widehat{U}^\top$ in terms of Δ. Let $Q^\perp := U_\perp U_\perp^\top = I_p - UU^\top$ denotes the orthogonal projection. For all $t \geq 1$, we define $Q^{-t} := U\Lambda^{-t}U^\top$. We slightly abuse the notation and denote $Q^0 := Q^\perp = U_\perp U_\perp^\top$.

LEMMA 3.1. *Suppose that $2\|\Delta\| \leq \lambda_r$, then*

$$\widehat{U}\widehat{U}^\top - UU^\top = \sum_{k \geq 1} \mathcal{S}_{\Sigma,k}(\Delta),$$

where the k-th order term $\mathcal{S}_{\Sigma,k}(\Delta)$ is a summation of $\binom{2k}{k}$ terms defined by

$$\mathcal{S}_{\Sigma,k}(\Delta) = \sum_{\mathbf{s}: s_1 + \ldots + s_{k+1} = k} (-1)^{1 + \tau(\mathbf{s})} \cdot Q^{-s_1} \Delta Q^{-s_2} \ldots \Delta Q^{-s_{k+1}},$$

where $\mathbf{s} = (s_1, \ldots, s_{k+1})$ contains non-negative indices and $\tau(\mathbf{s}) = \sum_{j=1}^{k+1} \mathbb{I}(s_j > 0)$. A sim-ple upper bound of the k-th order term is

$$\left\| \mathcal{S}_{\Sigma,k}(\Delta) \right\| \leq \binom{2k}{k} \left(\frac{\|\Delta\|}{\lambda_r} \right)^k.$$

Based on Lemma 3.1, the leading term, i.e., the 1st-order term, of $\widehat{U}\widehat{U}^\top - UU^\top$ is contributed by $\Lambda^{-1}U^\top \Delta U_\perp$ and $U_\perp^\top \Delta U \Lambda^{-1}$. The latter terms can be sharply controlled by exploiting the statistical properties of Δ if observations are i.i.d. sampled.

3.2. *Upper bounds.* In this section, we present the upper bounds of our (ε, δ)-DP estimator $\widetilde{U}\widetilde{U}^\top$ and $\widetilde{\Sigma}$. Let $\|\cdot\|_q$ denotes the matrix Schatten-q norm for any $q \in [1, \infty]$, e.g., the spectral norm $\|\cdot\|$ if $q = \infty$, the Frobenius norm $\|\cdot\|_F$ if $q = 2$, and the nuclear norm $\|\cdot\|_*$ if $q = 1$. A simple fact by the triangle inequality

$$\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_q \leq \|\widetilde{U}\widetilde{U}^\top - \widehat{U}\widehat{U}^\top\|_q + \|\widehat{U}\widehat{U}^\top - UU^\top\|_q,$$

leads to the following theorem.

THEOREM 3.2. *Suppose that $n \geq C_1(r \log n + \log^2 n), 2r + C_1 \log n \leq p$, and $\lambda/\sigma^2 \geq C_1(p/n + \sqrt{p/n})$ for some large absolute constant $C_1 > 0$. If we choose*

$$\Delta_1 = C_2 \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right) \frac{\sqrt{p(r + \log n)}}{n},$$

then, there exist absolute constants $c_1, C_4 > 0$ such that, for any $\varepsilon > 0, \delta \in (0,1)$, Algorithm 1 outputs an (ε, δ)-DP estimator $\widetilde{U}\widetilde{U}^\top$ satisfying

$$\frac{\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_q}{r^{1/q}} \leq C_4 \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right) \left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{r + \log n}}{n\varepsilon}\sqrt{\log \frac{2.5}{\delta}}\right),$$

with probability at least $1 - e^{-c_1 p} - 3n^{-9} - 10^{-20\tilde{r}}$. Moreover,

$$\frac{\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_q}{r^{1/q}} \leq C_4 \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right) \left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{r + \log n}}{n\varepsilon}\sqrt{\log \frac{2.5}{\delta}}\right).$$

Here, q can be any number in $[1, \infty]$ and $\tilde{r}$ denotes the effective rank of Σ.

Basically, the upper bounds consist of two parts: the first one represent the statistical error rate and the second one is the cost of privacy constraint. It is well-known that the first term is minimax optimal [10, 30, 35]. The second term decays at the rate $O\left(p/(n\varepsilon)\log^{1/2}\delta^{-1}\right)$ with respect to the sample size, dimension and privacy-related parameters, which is typical in differentially private algorithms [13, 34]. In Section 4, we shall develop matching minimax lower bounds showing that the rates in Theorem 3.2 are minimax optimal up the $\log n$ and $\log(2.5/\delta)$ terms.

It worth to mention that the $\log n$ term appearing in the privacy-related rate is due to the requirement of differential privacy that applies to each of the n observations. This $\log n$ term seems to be present in the upper bounds of most differentially private algorithms. See, e.g., [12, 13, 23] and references therein. A slight difference here is that the $\log n$ term appears not as an additional factor, but as an additive term. If $r \geq \log n$, the logarithmic factor can be ignored and the rate becomes minimax optimal except for the $\log \delta^{-1}$ factor. Note that the probability guarantee in Theorem 3.2 depends on the effective rank $\tilde{r}$. The rationale is that if the nuisance variance σ^2 is very small, e.g., $\sigma^2 = 0$, the distribution of $\mathcal{N}(0, U\Lambda U^\top + \sigma^2 I_p)$ becomes actually degenerate to a distribution in r-dimensional space. The concentration phenomenon for an r-dimensional distribution can be weaker than that for a p-dimensional distribution.

We now present the performance bound of our differentially private covariance estimator $\widetilde{\Sigma}$.

THEOREM 3.3. *Suppose that $n \geq C_1(r \log n + \log^2 n), 2r + C_1 \log n \leq p$, and $\lambda/\sigma^2 \geq C_1(p/n + \sqrt{p/n})$ for some large absolute constant $C_1 > 0$. If we choose*

$$\Delta_1 = C_2\left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\frac{\sqrt{p(r + \log n)}}{n} \quad \text{and} \quad \Delta_2 = C_2\frac{\lambda(r + \log n) + \sigma^2 p}{n},$$

then, there exist absolute constants $c_1, C_4 > 0$ such that, for any $\varepsilon > 0, \delta \in (0, 1)$, Algorithm 1 outputs an (ε, δ)-DP estimator $\widetilde{\Sigma}$ satisfying

$$\frac{\|\widetilde{\Sigma} - \Sigma\|_q}{r^{1/q}}$$

$$\leq C_4\left(\lambda\left(\sqrt{\frac{r}{n}} + \frac{\sqrt{r}(r + \log n)}{n\varepsilon} \cdot \sqrt{\log\frac{2.5}{\delta}}\right) + \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right)\right),$$

with probability at least $1 - 10^{-19r} - 3n^{-9} - e^{-c_1 p}$. Moreover,

$$\frac{\mathbb{E}\|\widetilde{\Sigma} - \Sigma\|_q}{r^{1/q}}$$

$$\leq C_4\left(\lambda\left(\sqrt{\frac{r}{n}} + \frac{\sqrt{r}(r + \log n)}{n\varepsilon} \cdot \sqrt{\log\frac{2.5}{\delta}}\right) + \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right)\right).$$

By Theorem 3.3, the privacy-irrelevant error rate

$$\lambda\sqrt{\frac{r}{n}} + \sqrt{\sigma^2(\lambda + \sigma^2)}\sqrt{\frac{p}{n}},$$

matches the minimax optimal rate of spiked covariance estimation in the existing literature [10, 15]. For ease of discussion, let us focus on the error rate in spectral norm. There are two terms related to the cost of privacy:

$$\lambda \cdot \frac{\sqrt{r}(r + \log n)}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}} \quad \text{and} \quad \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right),$$

where the second term is approximately of order $\lambda\|\widetilde{U}\widetilde{U}^\top - UU^\top\|$, contributed by the cost of estimating the eigenvectors. The first term grows at the rate $O(r^{3/2})$ with respect to the rank, which is contributed by the cost of estimating the eigenvalues. Due to the unknown orthogonal rotation measuring the alignment between $\widehat{U}$ and $\widetilde{U}$, privacy cost is also paid for the $r \times r$ unknown rotation matrix. Minimax lower bounds are developed in Section 4 demonstrating the optimality of these bound up to the $\log n$ and $\log(2.5/\delta)$ related terms.

4. Minimax Lower Bounds. In this section, we establish the minimax lower bound of PCA and covariance matrix estimation under the constraint of differential privacy. Our main technical tool is a version of Fano's lemma with privacy constraint.

4.1. *DP-constrained Fano's Lemma.* Several techniques have been developed to establish minimax lower bounds under the constraint of differential privacy. Notable examples include the fingerprint method [28], Le Cam's method under differential privacy [5], differentially private Fano's lemma [3], and the recently introduced Score Attack method [13]. Le Cam's method and Fano's lemma construct a multitude of hypotheses that are difficult to distinguish, while the fingerprint method and Score Attack design a test statistic with a prior distribution.

For our convenience, we employ the differentially private Fano's lemma, as detailed in Lemma 4.1, whose proof is provided in Section A.3 of the Supplementary Materials [14]. Here, $\mathrm{KL}(\cdot,\cdot)$ and $\mathrm{TV}(\cdot,\cdot)$ denote the Kullback-Leibler divergence and total variation distance between two distributions.

LEMMA 4.1. *Let* $\mathcal{P} := \{P : P = \mu^{(1)} \times \cdots \times \mu^{(n)}\}$ *be a family of product measures indexed by a parameter from a pseudo-metric space* (Θ, ρ). *Denote* $\theta(P) \in \Theta$ *the parameter associated with the distribution* P. *Let* $\mathcal{Q} = \{P_1, \cdots, P_N\} \subset \mathcal{P}$ *contain* N *probability measures and there exist constants* $\rho_0, l_0, t_0 > 0$ *such that for all* $i \neq i' \in [N]$,

$$\rho\left(\theta(P_i), \theta(P_{i'})\right) \geqslant \rho_0, \quad \mathrm{KL}\left(P_i \| P_{i'}\right) \leq l_0,$$

and

$$\sum_{k \in [n]} \mathrm{TV}\left(\mu_i^{(k)}, \mu_{i'}^{(k)}\right) \leq t_0,$$

where $P_i = \mu_i^{(1)} \times \cdots \times \mu_i^{(n)}$ *and* $P_{i'} = \mu_{i'}^{(1)} \times \cdots \times \mu_{i'}^{(n)}$. *Then,*

(8)

$$\inf_{A \in \mathcal{A}_{\varepsilon,\delta}(\mathcal{P})} \sup_{P \in \mathcal{P}} \mathbb{E}_A\, \rho(A, \theta(P)) \geqslant \max\left\{\frac{\rho_0}{2}\left(1 - \frac{l_0 + \log 2}{\log N}\right), \frac{\rho_0}{4}\left(1 \bigwedge \frac{N-1}{\exp\left(4\varepsilon t_0\right)}\right)\left(1 - \frac{2\delta e^{4\varepsilon t_0}}{e^\varepsilon - 1}\right)\right\},$$

where the infimum is taken over all the (ε, δ)-*DP randomized algorithm defined by* $\mathcal{A}_{\varepsilon,\delta}(\mathcal{P}) := \{A : X \mapsto \Theta \text{ and } A \text{ is } (\varepsilon, \delta)\text{-differentially private for all } X \sim P \in \mathcal{P}\}$.

Lemma 4.1 provides a powerful tool for developing a minimax lower bound in estimation problems under the constraint of differential privacy. Basically, if one can construct a sufficiently large set of distributions which are pairwise close in both Kullback-Leibler divergence and total variation distance, then a minimax lower bound can be derived if the underlying parameters are well-separated. The first term in the RHS of (8) is derived from the classic Fano's Lemma without privacy constraint and serves as a lower bound for the statistical error rate. This term is a well-established outcome in information theory by the framework of hypothesis testing and has been extensively employed in the statistics literature. The second term in the RHS of (8) characterizes the price one needs to pay for differential privacy. It is noteworthy that the cost of privacy is determined by t_0, which is the summation of marginal total variances. Intuitively, if the marginal total variance distances between $P_i = \mu_i^{(1)} \times \cdots \times \mu_i^{(n)}$ and $P_{i'} = \mu_{i'}^{(1)} \times \cdots \times \mu_{i'}^{(n)}$ are small , it becomes challenging to identify the distribution from which the dataset is drawn. Therefore, the cost of privacy is expected to be low when t_0 is small. Moreover, if we assume that $X = (X_1, \cdots, X_n) \sim P_i$, then the cost of privacy resulting from replacing $X_k \sim \mu_i^{(k)}$ by $X_k' \sim \mu_{i'}^{(k)}$ should be upper bounded in terms of $\mathrm{TV}(\mu_i^{(k)}, \mu_{i'}^{(k)})$.

4.2. *Minimax lower bounds.* In this section, we apply Lemma 4.1 to establish the minimax lower bounds for differentially private PCA and covariance estimation under the spiked covariance model. Denote the family of normal distribution with a spiked covariance matrix by

$$\mathcal{P}(\lambda, \sigma^2) := \left\{\mathcal{N}(0, \Sigma) : \Sigma = U\Lambda U^\top + \sigma^2 I_p \in \Theta(\lambda, \sigma^2)\right\}.$$

By definition, each distribution $P \in \mathcal{P}(\lambda, \sigma^2)$ is indexed by the pair of eigenvalues Λ and eigenvectors $U \in \mathbb{O}_{p,r}$. We first focus on the minimax lower bounds for estimating the spectral projector $UU^\top$. Similarly, the minimax lower bounds are established in all Schatten-q norms for $q \in [1, \infty]$.

THEOREM 4.2. *Let the $p \times n$ data matrix X have i.i.d. columns sampled from a distribution $P = \mathcal{N}(0, U^\top \Lambda U^\top + \sigma^2 I_p) \in \mathcal{P}(\lambda, \sigma^2)$. Then, there exists an absolute constant $c_1 > 0$ such that*

$$\inf_{\widetilde{U} \in \mathcal{U}_{\varepsilon,\delta}} \sup_{P \in \mathcal{P}(\lambda,\sigma^2)} \frac{\mathbb{E}\|\tilde{U}\tilde{U}^\top - UU^\top\|_q}{r^{1/q}} \geq c_1 \left(\left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}} \right) \left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{r}}{n\epsilon} \right) \right) \bigwedge 1,$$

where the infimum is taken over all the possible (ε, δ)-DP algorithms, denoted by $\mathcal{U}_{\varepsilon,\delta}$, and the expectation is taken with respect to both $\widetilde{U}$ and P.

It is worth noting the two terms in the minimax lower bound of spectral norm ($q = \infty$):

$$(9) \qquad \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}} \right) \sqrt{\frac{p}{n}} \quad \text{and} \quad \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}} \right) \frac{p\sqrt{r}}{n\varepsilon}.$$

The first term concerns the statistical error of PCA without privacy constraint. The error bound is free of the rank r, which is very typical in spectral norm error rate and the rate matches the existing minimax optimal rate of PCA for spiked covariance model. See, e.g., [10, 49, 50]. The second term is the price paid for differential privacy. Interestingly, the second term is dependent on the rank r even though spectral norm is considered here. The technical explanation is that the sensitivity of empirical spectral projector increases as the number of PC's grows. Comparing the two terms in (9), we observe that if $\varepsilon \geq (rp/n)^{1/2}$, the cost of privacy is dominated by the statistical error.

A minimax lower bound for *rank-one* PCA has been established in [34, Theorem 5.3]. Their developed rate in spectral norm also have two terms:

$$\sqrt{\frac{\sigma^2}{\lambda + \sigma^2}} \cdot \sqrt{\frac{p}{n}} \quad \text{and} \quad \sqrt{\frac{\sigma^2}{\lambda + \sigma^2}} \cdot \frac{p}{n\varepsilon}.$$

Their rate matches ours when $r = 1$ and $\lambda \geq \sigma^2$. On the other hand, if $\lambda \ll \sigma^2$, our minimax lower bound is much stronger. Moreover, our minimax lower bounds hold for a diverging rank as long as $2r \leq p$.

We now shift our focus to the minimax lower bound of differentially private estimation of the spiked covariance matrix. Here, we assume σ^2 is known and it suffices to estimate the signal part $U\Lambda U^\top$. As a result, the minimax lower bound is essentially determined jointly by the lower bounds in estimating eigenvalues and eigenvectors.

THEOREM 4.3. *Let the $p \times n$ data matrix X have i.i.d. columns sampled from a distribution $P = \mathcal{N}(0, U^\top \Lambda U^\top + \sigma^2 I_p) \in \mathcal{P}(\lambda, \sigma^2)$. Then, there exists an absolute constant $c_1 > 0$ such that*

$$\inf_{\widetilde{\Sigma} \in \mathcal{M}_{\varepsilon,\delta}} \sup_{P \in \mathcal{P}(\lambda,\sigma^2)} \frac{\mathbb{E}\|\widetilde{\Sigma} - \Sigma\|_q}{r^{1/q}} \geq c_1 \left(\lambda \left(\sqrt{\frac{r}{n}} + \frac{r^{3/2}}{n\varepsilon} \right) + \sqrt{\sigma^2(\lambda + \sigma^2)} \left(\sqrt{\frac{p}{n}} + \frac{\sqrt{r}p}{n\varepsilon} \right) \right) \bigwedge \lambda,$$

where the infimum is taken over all the possible (ε, δ)-DP algorithms, denoted by $\mathcal{M}_{\varepsilon,\delta}$, and the expectation is taken with respect to both $\widetilde{\Sigma}$ and P. Here, q can be any number in $[1, \infty]$.

Without loss of generality, let us discuss the two terms in the spectral norm distance

$$(10) \qquad \lambda \left(\sqrt{\frac{r}{n}} + \frac{r^{3/2}}{n\varepsilon} \right) \quad \text{and} \quad \sqrt{\sigma^2(\lambda + \sigma^2)} \left(\sqrt{\frac{p}{n}} + \frac{\sqrt{r}p}{n\varepsilon} \right).$$

The second term is contributed by the differentially private estimation error of PCA in the form of $\lambda\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_F^2$. The first term dominates if the signal strength is exceedingly

large, or more precisely, when $\lambda/\sigma^2 \gg p/r$. In this case, we can simply regard $\sigma = 0$ and the stochastic error mainly comes from the randomness of a low-dimensional distribution. Basically, it suffices to consider the minimax optimal estimation under a smaller family of normal distributions $\{\mathcal{N}(0, \lambda UU^\top + \lambda I_r) : U \in \mathbb{O}_{r,r/4}\}$. By replacing $\sigma \leftarrow \lambda$, $r \leftarrow r/4$, and $p \leftarrow r$, the second term reduces to the first term in (10). Without the privacy constraint, the first term also matches the existing optimal rate in covariance estimation under spiked covariance model [10, 15].

5. Extensions. For the sake of clarity, we have assumed uniformity in the order of spiked eigenvalues and Gaussian distributions. In this section, we extend our analysis to provide upper bounds for differentially private PCA and covariance estimation without requiring these specific conditions.

5.1. *Diverging condition number.* Suppose that $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \Sigma)$ where $\Sigma = U\Lambda U^\top + \sigma^2 I_p$ with spiked eigenvalues $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_r)$. Denote $\kappa_0 := \lambda_1/\lambda_r$, the ratio of the largest and smallest spiked eigenvalues. The proof of Corollary 5.1 is almost identical to that of Theorems 3.2 and 3.3, and thus omitted. We only present the upper bounds of the expected error in Schatten norms, but high probability bounds hold similarly.

COROLLARY 5.1. *Suppose that $n \geq C_1(\kappa_0^2 r \log n + \log^2 n), 2r + C_1 \log n \leq p$, and $\lambda_r/\sigma^2 \geq C_1(\kappa_0 p/n + \sqrt{p/n})$ for some large absolute constant $C_1 > 0$. If we choose*

$$\Delta_1 = C_2 \left(\frac{\sigma^2}{\lambda_r} + \sqrt{\frac{\kappa_0 \sigma^2}{\lambda_r}} \right) \frac{\sqrt{p(r + \log n)}}{n} \quad \text{and} \quad \Delta_2 = C_2 \frac{\lambda_1(r + \log n) + \sigma^2 p}{n},$$

then, there exist absolute constants $C_4 > 0$ such that, for any $\varepsilon > 0, \delta \in (0,1)$, Algorithm 1 outputs an (ε, δ)-DP estimators $\widetilde{U}\widetilde{U}^\top$ and $\widetilde{\Sigma}$ satisfying

$$\frac{\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_q}{r^{1/q}} \leq C_4 \left(\frac{\sigma^2}{\lambda_r} + \sqrt{\frac{\kappa_0 \sigma^2}{\lambda_r}} \right) \left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{r + \log n}}{n\varepsilon} \log^{1/2} \left(\frac{2.5}{\delta} \right) \right).$$

and

$$\frac{\mathbb{E}\|\widetilde{\Sigma} - \Sigma\|_q}{r^{1/q}}$$

$$\leq C_4 \left(\lambda_1 \left(\sqrt{\frac{r}{n}} + \frac{\sqrt{r}(r + \log n)}{n\varepsilon} \cdot \sqrt{\log \frac{2.5}{\delta}} \right) + \sqrt{\sigma^2(\lambda_1 + \sigma^2)} \left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon} \sqrt{\log \frac{2.5}{\delta}} \right) \right).$$

for all $q \in [1, \infty]$.

5.2. *Sub-Gaussian.* Suppose that X follows a sub-Gaussian distribution satisfying that, for any $u \in \mathbb{R}^p$, the following bound holds

$$\mathbb{E} \exp \left\{ \frac{\langle X, u \rangle^2}{u^\top \Sigma u} \right\} \leq 2,$$

where $\Sigma \in \Theta(\lambda, \sigma^2)$. For ease of exposition, we focus on the case of bounded condition number. Interestingly, the sensitivity of eigenvectors and eigenvalues is actually identical to that under Gaussian distributions.

COROLLARY 5.2. *Suppose that $n \geq C_1\big(r\log(p+n)\log^2 r + \log^2 n\big)$, $2r + C_1\log n \leq p$, and $\lambda/\sigma^2 \geq C_1(p/n + \sqrt{p/n})\log(p+n)$ for some large absolute constant $C_1 > 0$. If we choose*

$$\Delta_1 = C_2\bigg(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\bigg)\frac{\sqrt{p(r+\log n)}}{n} \quad \text{and} \quad \Delta_2 = C_2\frac{\lambda(r+\log n) + \sigma^2 p}{n},$$

then, there exist absolute constant $C_4 > 0$ such that, for any $\varepsilon > 0, \delta \in (0,1)$, Algorithm 1 outputs an (ε, δ)-DP estimators $\widetilde{U}\widetilde{U}^\top$ and $\widetilde{\Sigma}$ satisfying

$$\frac{\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_q}{r^{1/q}} \leq C_4\bigg(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\bigg)\bigg(\sqrt{\frac{p\log p}{n}} + \frac{p\sqrt{r+\log n}}{n\varepsilon}\log^{1/2}\Big(\frac{2.5}{\delta}\Big)\bigg),$$

and

$$\frac{\mathbb{E}\|\widetilde{\Sigma} - \Sigma\|_q}{r^{1/q}}$$

$$\leq C_4\bigg(\lambda\bigg(\sqrt{\frac{r}{n}} + \frac{\sqrt{r}(r+\log n)}{n\varepsilon}\cdot\sqrt{\log\frac{2.5}{\delta}}\bigg) + \sqrt{\sigma^2(\lambda+\sigma^2)}\bigg(\sqrt{\frac{p\log p}{n}} + \frac{p\sqrt{r+\log n}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\bigg)\bigg).$$

for all $q \in [1,\infty]$.

As shown by Corollary 5.2, the upper bounds of differentially private sub-Gaussian PCA and covariance estimation are almost the same as those for Gaussian distributions, implying that these bounds are minimax optimal. However, some additional logarithmic factors appear in the upper bound and signal-to-noise ratio condition when controlling the higher-order terms in spectral perturbation.

6. Proofs of Main Results. In this section, we prove the main results and some of the key lemmas. Some additional technical results are given in the Supplementary Materials [14]. We begin by stating several technical lemmas that will be frequently used in the subsequent proofs. Due to space constraints, their proofs are given in the supplement [14].

6.1. *Technical lemmas.* Lemma 6.1 is a well-known *dimension-free* concentration inequality of sample covariance matrix developed by [30]. Here, $\|\cdot\|$ denote the spectral norm of a matrix and ℓ_2-norm of a vector.

LEMMA 6.1 ([30]). *Suppose $X_1,\cdots,X_n$ are i.i.d. sampled from $\mathcal{N}(0,\Sigma)$ and $\widehat{\Sigma} := \sum_{i=1}^n X_i X_i^\top/n$. Then,*

$$\mathbb{E}\|\widehat{\Sigma} - \Sigma\| \asymp \bigg(\sqrt{\frac{\mathrm{tr}(\Sigma)\|\Sigma\|}{n}} \bigvee \frac{\mathrm{tr}(\Sigma)}{n}\bigg).$$

Moreover, there exists an absolute constant $C_1 > 0$ such that, for all $t \geq 1$, with probability at least $1 - e^{-t}$,

$$\Big|\|\widehat{\Sigma} - \Sigma\| - \mathbb{E}\|\widehat{\Sigma} - \Sigma\|\Big| \leq C_1\|\Sigma\|\bigg(\frac{t}{n} + \sqrt{\frac{t}{n}}\bigg(1 + \sqrt{\frac{\mathrm{tr}(\Sigma)/\|\Sigma\|}{n}}\bigg)\bigg).$$

The following lemma characterizes the concentration of the norm of a Gaussian random vector.

LEMMA 6.2. *Let $X \sim \mathcal{N}(0, \Sigma)$ and the eigenvalues of Σ are $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$. Then, there exist absolute constants $C_1, C_2, c_1 > 0$ such that*

$$\mathbb{P}\left(\left| \|X\|^2 - \mathrm{tr}(\Sigma) \right| \leq C_1 \left(u \sum_{i=1}^{p} \lambda_i^2 \right)^{1/2} + C_2 \lambda_1 u \right) \geq 1 - e^{-c_1 u},$$

for any $u > 0$. Under the spiked covariance model $\Sigma \in \Theta(\lambda, \sigma^2)$ and the condition that $p \geq C_6 \log n$ for some absolute constant $C_6 > 0$, we have

$$\mathbb{P}\Bigg(\Big\{ \max_{i \in [n]} \|X_i\|^2 + \|X_i'\|^2 \leq C_3(r\lambda + p\sigma^2) + C_4\sqrt{(r\lambda^2 + p\sigma^4)\log n} + C_5(\lambda + \sigma^2)\log n \Big\}$$

$$\bigcap \Big\{ \max_{i \in [n]} \|U^\top X_i\|^2 + \max_{i \in [n]} \|U^\top X_i'\|^2 \leq C_3 r(\lambda + \sigma^2) + C_4\sqrt{r(\lambda^2 + \sigma^4)\log n} + C_5(\lambda + \sigma^2)\log n \Big\}$$

$$\bigcap \Big\{ \max_{i \in [n]} \|U_\perp^\top X_i\|^2 + \max_{i \in [n]} \|U_\perp^\top X_i'\|^2 \leq C_3 p\sigma^2 \Big\} \geq 1 - n^{-10},$$

where $C_3, C_4, C_5 > 0$ are some absolute constants. Let $\mathcal{E}_0$ denote the above event. Moreover,

$$\mathbb{E}\|X_i\|^2 \leq C_3(r\lambda + p\sigma^2), \quad \mathbb{E}\|U^\top X_i\|^2 \leq C_3 r(\lambda + \sigma^2) \quad \text{and} \quad \mathbb{E}\|U_\perp^\top X_i\|^2 \leq C_3 p\sigma^2.$$

Denote $\Delta := \widehat{\Sigma} - \Sigma$ and $\Delta^{(i)} := \widehat{\Sigma}^{(i)} - \Sigma$. We shall frequently use several concentration bounds related to Δ and $\Delta^{(i)}$ throughout the proof. For reader's convenience, these concentration bounds are collected in the following lemma. Recall that $\tilde{r} = \mathrm{tr}(\Sigma)/\|\Sigma\|$ is the effective rank of Σ.

LEMMA 6.3. *Suppose that $\Sigma \in \Theta(\lambda, \sigma^2)$, $n \geq C_1(r + \log^2 n)$, $2r + C_1 \log n \leq p$, and $\lambda/\sigma^2 \geq C_1 p/n$ for some absolute constant $C_1 > 0$. There exist absolute constants $C_2 > 0$ such that the event*

$$(11) \qquad \mathcal{E}_\Delta := \left\{ \|\Delta\| + \max_{i \in [n]} \|\Delta^{(i)}\| \leq C_2 \sqrt{\frac{(\lambda + \sigma^2)(r\lambda + p\sigma^2)}{n}} \right\}$$

holds with probability $\mathbb{P}(\mathcal{E}_\Delta) \geq 1 - n^{-10} - 10^{-20\tilde{r}}$. Meanwhile, the upper bound in (11) also holds for $\mathbb{E}\|\Delta\|$. There exist absolute constants $c_2, C_3 > 0$ such that the event

$$(12)$$

$$\mathcal{E}_1 := \left\{ \|U^\top \Delta U_\perp\| + \max_{i \in [n]} \|U^\top \Delta^{(i)} U_\perp\| \leq C_3 \sqrt{\frac{\sigma^2(\lambda + \sigma^2)p}{n}} \right\}$$

$$\bigcap \left\{ \max_{i \in [n]} \|U^\top (X_i X_i^\top / n) U_\perp\| + \|U^\top (X_i' X_i'^\top / n) U_\perp\| \leq C_3 \frac{\sqrt{\sigma^2(\lambda + \sigma^2)p(r + \log n)}}{n} \right\}$$

$$\bigcap \left\{ \max_{i \in [n]} \left\| U^\top \Big(\frac{1}{n} \sum_{j \neq i} X_j X_j^\top \Big) U_\perp U_\perp^\top X_i \right\| \leq C_3 \sigma \cdot \sqrt{\frac{\sigma^2(\lambda + \sigma^2)p(r + \log n)}{n}} \right\}$$

holds with probability $\mathbb{P}(\mathcal{E}_1) \geq 1 - e^{-c_1 p} - 2n^{-9}$. Meanwhile, the first upper bound in (12) also hods for $\mathbb{E}\|U^\top \Delta U_\perp\|$.

The following perturbation bound of principal subspace will be useful.

LEMMA 6.4. *Suppose that $\lambda_r \geq (4 + \delta)\|\Delta\|$ for some $\delta > 0$, then*

$$\left\| \widehat{U}\widehat{U}^\top - UU^\top \right\| \leq 2 \left\| \Lambda^{-1} U^\top \Delta U_\perp \right\| + 6(4 + \delta) \frac{\|\Delta\| \left\| U^\top \Delta U_\perp \right\|}{\delta \lambda_r^2}.$$

6.2. *Proof of Lemma 2.3.* Let the events $\mathcal{E}_0, \mathcal{E}_\Delta, \mathcal{E}_1$ be defined as in Section 6.1. The following analysis proceeds mainly on the event $\mathcal{E}^* := \mathcal{E}_0 \cap \mathcal{E}_\Delta \cap \mathcal{E}_1$, which occurs with probability at least $1 - e^{-c_1 p} - 3n^{-9} - 10^{-20\tilde{r}}$.

On the event $\mathcal{E}^*$ and under the conditions that $\Sigma \in \Theta\left(p, r, \lambda, \sigma^2\right)$, $n \geq C_1(r \log n + \log^2 n)$, $2r + C_1 \log n \leq p$ and $\lambda/\sigma^2 \geq C_1\left(p/n + \sqrt{p/n}\right)$ for a large absolute constant $C_1 > 0$, we have $\lambda_r \geq 5\left(\|\Delta\| \vee \|\Delta^{(i)}\|\right)$. Therefore, we are able to apply Lemma 3.1 to obtain

$$\widehat{U}\widehat{U}^\top - UU^\top = \sum_{k \geq 1} \mathcal{S}_{\Sigma,k}(\Delta) \quad \text{and} \quad \widehat{U}^{(i)}\widehat{U}^{(i)\top} - UU^\top = \sum_{k \geq 1} \mathcal{S}_{\Sigma,k}(\Delta^{(i)}).$$

The explicit formula of spectral projectors implies that
(13)
$$\|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_{\mathrm{F}} = \left\|\sum_{k \geq 1} \mathcal{S}_{\Sigma,k}(\Delta) - \sum_{k \geq 1} \mathcal{S}_{\Sigma,k}(\Delta^{(i)})\right\|_{\mathrm{F}}$$

$$\leq \|\mathcal{S}_{\Sigma,1}(\Delta) - \mathcal{S}_{\Sigma,1}(\Delta^{(i)})\|_{\mathrm{F}} + \left\|\sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta) - \sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta^{(i)})\right\|_{\mathrm{F}}.$$

We now bound the first-order term $\|\mathcal{S}_{\Sigma,1}(\Delta) - \mathcal{S}_{\Sigma,1}(\Delta^{(i)})\|_{\mathrm{F}}$ and the higher order term $\|\sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta) - \sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta^{(i)})\|_{\mathrm{F}}$, separately.

Step 1: bounding the first order term. By the definitions of $\mathcal{S}_{\Sigma,1}(\Delta)$ and $\mathcal{S}_{\Sigma,1}(\Delta^{(i)})$,

$$\max_{i \in [n]} \|\mathcal{S}_{\Sigma,1}(\Delta) - \mathcal{S}_{\Sigma,1}(\Delta^{(i)})\|_{\mathrm{F}} \leq \max_{i \in [n]} \|Q^{-1}(\Delta - \Delta^{(i)})Q^\perp\|_{\mathrm{F}} + \|Q^\perp(\Delta - \Delta^{(i)})Q^{-1}\|_{\mathrm{F}}$$

$$\leq \frac{2}{n} \max_{i \in [n]} \left(\|\Lambda^{-1}U^\top X_i X_i^\top U_\perp\| + \|\Lambda^{-1}U^\top X_i' X_i'^\top U_\perp\|\right)$$

(14)
$$\leq C_3 \sqrt{\frac{\sigma^2(\lambda + \sigma^2)}{\lambda^2}} \cdot \frac{\sqrt{p(r + \log n)}}{n},$$

where the last inequality holds on $\mathcal{E}^*$ for all $i \in [n]$ based on Lemma 6.3.

Step 2: bounding the higher-order terms. Let I_k be the index set for terms in $\mathcal{S}_{\Sigma,k}$

$$I_k = \left\{\mathbf{s} : \mathbf{s} = (s_1, \ldots, s_{k+1}), \sum_{m=1}^{k+1} s_m = k, s_m \geq 0, \quad \forall m \in [k+1]\right\},$$

with the cardinality $|I_k| = \binom{2k}{k}$. We define

$$\mathcal{T}_{\Sigma,k,\mathbf{s},l}(\Delta - \Delta^{(i)}) := Q^{-s_1}\Delta^{(i)}Q^{-s_2}\cdots Q^{-s_l}(\Delta - \Delta^{(i)})Q^{s_{l+1}}\cdots Q^{-s_k}\Delta Q^{s_{k+1}},$$

for $k \geq 2$, $\mathbf{s} = (s_1, \cdots, s_{k+1}) \in I_k$ and $l \in [k]$. Since $|I_k| = \binom{2k}{k}$, the higher order terms can be bounded as follows

$$\left\|\sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta) - \sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta^{(i)})\right\|_{\mathrm{F}} = \left\|\sum_{k \geq 2} \sum_{\mathbf{s} \in I_k} \sum_{l \in [k]} \mathcal{T}_{\Sigma,k,\mathbf{s},l}(\Delta - \Delta^{(i)})\right\|_{\mathrm{F}}$$

(15)
$$\leq \sum_{k \geq 2} \binom{2k}{k} \max_{\mathbf{s} \in I_k} \sum_{l \in [k]} \|\mathcal{T}_{\Sigma,k,\mathbf{s},l}(\Delta - \Delta^{(i)})\|_{\mathrm{F}}.$$

It suffices to upper bound $\|\mathcal{T}_{\Sigma,k,\mathbf{s},l}(\Delta - \Delta^{(i)})\|_{\mathrm{F}}$ for any $\mathbf{s} \in I_k$. Denote

$$D_{\max} := C_2 \sqrt{\frac{(\lambda + \sigma^2)(r\lambda + p\sigma^2)}{n}},$$

the upper bound appeared in the event $\mathcal{E}_\Delta$ so that $\|\Delta\| + \max_{i \in [n]} \|\Delta^{(i)}\| \leq D_{\max}$ in the event $\mathcal{E}_\Delta$.

LEMMA 6.5.   *Under the conditions of Lemma 2.3, the following bound holds in the event $\mathcal{E}^*$ for all $k \geq 2$ and $\mathbf{s} \in I_k$.*

$$\sum_{l \in [k]} \|\mathcal{T}_{\Sigma,k,\mathbf{s},l}(\Delta - \Delta^{(i)})\|_{\mathrm{F}} \leq C_6 \frac{(3+k)k}{2} \left(\frac{D_{\max}}{\lambda_r}\right)^{k-2} \cdot \frac{\sigma^2}{\lambda}\left(\sqrt{\frac{p}{n}} + \frac{p}{n}\right) \cdot \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}} \frac{\sqrt{p(r+\log n)}}{n},$$

where $C_6 > 0$ is an absolute constant.

We now continue from (15) and get for all $i \in [n]$

$$\left\|\sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta) - \sum_{k \geq 2} \mathcal{S}_{\Sigma,k}(\Delta^{(i)})\right\|_{\mathrm{F}} \leq \sum_{k \geq 2} \binom{2k}{k} \max_{\mathbf{s} \in I_k} \sum_{l \in [k]} \|\mathcal{T}_{\Sigma,k,\mathbf{s},l}(\Delta - \Delta^{(i)})\|_{\mathrm{F}}$$

$$\lesssim \sum_{k \geq 2} \binom{2k}{k} \frac{(3+k)k}{2} \left(\frac{D_{\max}}{\lambda_r}\right)^{k-2} \cdot \frac{\sigma^2}{\lambda}\left(\sqrt{\frac{p}{n}} + \frac{p}{n}\right) \cdot \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}} \frac{\sqrt{p(r+\log n)}}{n}$$

$$(16) \qquad \lesssim \frac{\sigma^2}{\lambda}\left(\sqrt{\frac{p}{n}} + \frac{p}{n}\right) \cdot \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}} \frac{\sqrt{p(r+\log n)}}{n},$$

where the last inequality holds if $\lambda \geq C_7 D_{\max}$ for a large enough constant $C_7 > 0$.

Combining (14) and (16) together with the condition $\lambda/\sigma^2 \geq C_3(\sqrt{p/n} + p/n)$, we get in event $\mathcal{E}^*$ that

$$\max_{i \in [n]} \|\widehat{U}\widehat{U}^\top - \widehat{U}^{(i)}\widehat{U}^{(i)\top}\|_{\mathrm{F}} \lesssim \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}} \frac{\sqrt{p(r+\log n)}}{n}.$$

6.3.   *Proof of Lemma 2.4.*   Recall that, by definition of Δ and $\Delta^{(i)}$, we can write

$$\widehat{\Sigma}^{(i)} = \widehat{\Sigma} - \frac{1}{n}X_i X_i^\top + \frac{1}{n}X_i' X_i'^\top.$$

By Hoffman-Weilandt's inequality, we have

$$\sum_{k=1}^{p}\left(\lambda_k(\widehat{\Sigma}) - \lambda_k(\widehat{\Sigma}^{(i)})\right)^2 \leq \|\widehat{\Sigma} - \widehat{\Sigma}^{(i)}\|_{\mathrm{F}}^2 \leq 2\|\widehat{\Sigma} - \widehat{\Sigma}^{(i)}\|^2 \leq \frac{4}{n^2}\left(\|X_i\|^2 + \|X_i'\|^2\right)^2$$

$$\leq C_4\left(\frac{\lambda(r+\log n) + p\sigma^2}{n}\right)^2,$$

where the last inequality holds in the event $\mathcal{E}_0$ defined in Lemma 6.2 and under the condition $p \geq C_1 \log n$. This completes the proof.

6.4.   *Proof of Theorem 3.2.*   It suffices to bound $\|\widetilde{U}\widetilde{U}^\top - \widehat{U}\widehat{U}^\top\|_q$ and $\|\widehat{U}\widehat{U}^\top - UU^\top\|_q$. Without loss of generality, we start with $q = \infty$, i.e., the upper bound of spectral norm.

Recall that $\widetilde{U}\widetilde{U}^\top$ denotes the spectral projector of the top-r eigenvectors of $\Sigma + \Delta$. Based on Lemma 6.3,

$$\|\Delta\| \leq C_2\sqrt{\frac{(\lambda+\sigma^2)(r\lambda + p\sigma^2)}{n}}$$

with probability at least $1 - n^{-10} - 10^{-20\tilde{r}}$. In this event, we have $\lambda \geq 5\|\Delta\|$ since $n \geq C_1 r$ and $\lambda/\sigma^2 \geq C_1(p/n + \sqrt{p/n})$. Applying Lemma 6.4, we get in this event,

$$\left\|\widehat{U}\widehat{U}^\top - UU^\top\right\| \leq 2\left\|\Lambda^{-1}U^\top\Delta U_\perp\right\| + C_3\frac{\|\Delta\|\left\|U^\top\Delta U_\perp\right\|}{\lambda^2} \leq C_4\left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\sqrt{\frac{p}{n}},$$

where the last inequality holds with probability at least $1 - 2n^{-9} - e^{-c_1 p}$ by Lemma 6.3. Moreover,

$$\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\| \leq C_4\left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\sqrt{\frac{p}{n}}\bigwedge 1.$$

By Davis-Kahan theorem, $\|\widetilde{U}\widetilde{U}^\top - \widetilde{U}\widetilde{U}^\top\| \lesssim \|Z\| \wedge 1$, where Z is a symmetric matrix with i.i.d. entries having distribution $\mathcal{N}\left(0, 8\Delta_1^2\varepsilon^{-2}\log\frac{2.5}{\delta}\right)$. By [45, Theorem 4.4.5],

$$\|Z\| \lesssim \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\cdot\frac{p\sqrt{r + \log n}}{n\varepsilon}\log^{1/2}(\frac{2.5}{\delta}),$$

with probability at least $1 - O\left(e^{-c_1 p}\right)$ and the same bound also holds for $\mathbb{E}\|Z\|$.

Combining the above two bounds, we can conclude that with probability at least $1 - e^{-c_1 p} - 3n^{-9} - 10^{-20\tilde{r}}$,

$$\|\widetilde{U}\widetilde{U}^\top - UU^\top\| \lesssim \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\cdot\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right),$$

and the same bound also holds for $\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\|$. In the same event, we can also get the upper bound of nuclear norm distance:

$$\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_* \lesssim \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\cdot r\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right)$$

and $\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_*$. For general $q \in [1, \infty]$, we can apply the interpolation inequality:

$$\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_q \leq \|\widetilde{U}\widetilde{U}^\top - UU^\top\|_*^{1/q}\cdot\|\widetilde{U}\widetilde{U}^\top - UU^\top\|^{\frac{q-1}{q}},$$

which completes the proof.

6.5. *Proof of Theorem 3.3.* Recall that we assume σ^2 is known. By the definition of $\widetilde{\Sigma}$ stated in Algorithm 1, for any $q \in [1, \infty]$, we have the Schatten-q norm bounded as

$$\|\widetilde{\Sigma} - \Sigma\|_q = \|\widetilde{U}\widetilde{\Lambda}\widetilde{U}^\top - U\Lambda U^\top\|_q$$

$$= \left\|\widetilde{U}\left(\widetilde{U}^\top(\widehat{\Sigma} - \sigma^2 I_p + E)\widetilde{U}\right)\widetilde{U}^\top - UU^\top(\Sigma - \sigma^2 I_p)UU^\top\right\|_q$$

$$\leq \left\|\widetilde{U}\widetilde{U}^\top(\widehat{\Sigma} - \sigma^2 I_p)\widetilde{U}\widetilde{U}^\top - UU^\top(\Sigma - \sigma^2 I_p)UU^\top\right\|_q + \|E\|_q.$$

Without loss of generality, we begin with $q = \infty$ and bound the spectral norm. Since E is an $r \times r$ symmetric matrix with i.i.d. entries $\mathcal{N}\left(0, 8\Delta_2^2\varepsilon^{-2}\log\left(\frac{2.5}{\delta}\right)\right)$, by [45, Theorem 4.4.5], we get

(17) $$\|E\| \leq C_4\frac{\lambda\sqrt{r}(r + \log n) + \sigma^2 p\sqrt{r}}{n\varepsilon}\cdot\sqrt{\log\frac{2.5}{\delta}},$$

with probability at least $1 - 10^{-20r}$ and for some absolute constant $C_4 > 0$. Moreover, the same bound holds for $\mathbb{E}\|E\|$.

Observe that

$$\big\|\widetilde{U}\widetilde{U}^\top(\widehat{\Sigma} - \sigma^2 I_p)\widetilde{U}\widetilde{U}^\top - UU^\top(\Sigma - \sigma^2 I_p)UU^\top\big\|$$

(18)
$$\leq \big\|\big(\widetilde{U}\widetilde{U}^\top - UU^\top\big)\big\|\big\|(\widehat{\Sigma} - \sigma^2 I_p)\big\| + \|UU^\top(\widehat{\Sigma} - \Sigma)\| + \|\Sigma - \sigma^2 I_p\|\|\widetilde{U}\widetilde{U}^\top - UU^\top\|.$$

Since

$$\|\widehat{\Sigma} - \sigma^2 I_p\| \leq \|\Sigma - \sigma^2 I_p\| + \|\widehat{\Sigma} - \Sigma\| \lesssim \lambda,$$

where the last inequality holds in event $\mathcal{E}_\Delta$ defined in Lemma 6.3 and under the conditions $n \geq C_1 r$ and $\lambda/\sigma^2 \geq C_1\big(p/n + \sqrt{p/n}\big)$. By (18), we get

$$\big\|\widetilde{U}\widetilde{U}^\top(\widehat{\Sigma} - \sigma^2 I_p)\widetilde{U}\widetilde{U}^\top - UU^\top(\Sigma - \sigma^2 I_p)UU^\top\big\|$$

$$\lesssim \lambda\|\widetilde{U}\widetilde{U}^\top - UU^\top\| + \|U^\top(\widehat{\Sigma} - \Sigma)\|$$

$$\lesssim \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right) + \sqrt{\frac{(\lambda + \sigma^2)(r\lambda + p\sigma^2)}{n}}$$

(19)
$$\lesssim \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right) + \lambda\sqrt{\frac{r}{n}},$$

where the last inequality is due to Theorem 3.2 and Lemma 6.3. Combining (17) and (19), we get

$$\|\widetilde{\Sigma} - \Sigma\| \lesssim \lambda\left(\sqrt{\frac{r}{n}} + \frac{\sqrt{r}(r + \log n)}{n\varepsilon} \cdot \sqrt{\log\frac{2.5}{\delta}}\right) + \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right),$$

with probability at least $1 - 10^{-20r} - 3n^{-9} - 10^{-20\tilde{r}} - e^{-c_1 p}$. Since $\tilde{r} \geq r$, we can simplify $10^{-20r} + 10^{-20\tilde{r}} \leq 10^{-19r}$.

Moreover,

$$\mathbb{E}\big\|\widetilde{U}\widetilde{U}^\top(\widehat{\Sigma} - \sigma^2 I_p)\widetilde{U}\widetilde{U}^\top - UU^\top(\Sigma - \sigma^2 I_p)UU^\top\big\|$$

$$\lesssim \mathbb{E}\big\|\big(\widetilde{U}\widetilde{U}^\top - UU^\top\big)\big\|\big\|(\widehat{\Sigma} - \sigma^2 I_p)\big\| + \mathbb{E}\|UU^\top(\widehat{\Sigma} - \Sigma)\| + \mathbb{E}\|\Sigma - \sigma^2 I_p\|\|\widetilde{U}\widetilde{U}^\top - UU^\top\|$$

$$\lesssim \mathbb{E}^{1/2}\big\|\big(\widetilde{U}\widetilde{U}^\top - UU^\top\big)\big\|^2 \mathbb{E}^{1/2}\big\|(\widehat{\Sigma} - \sigma^2 I_p)\big\|^2 + \sqrt{\frac{(\lambda + \sigma^2)(r\lambda + p\sigma^2)}{n}}$$

$$+ \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right).$$

Note that

$$\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\|^2 \leq 2\mathbb{E}\|\widetilde{U}\widetilde{U} - \widetilde{U}\widetilde{U}^\top\|^2 + 2\mathbb{E}\|\widetilde{U}\widetilde{U}^\top - UU^\top\|^2$$

$$\lesssim \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right) \cdot \left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{r + \log n}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right),$$

where the last inequality is due to the classical concentration of operator norm of a Gaussian random matrix, e.g., [32]. and

$$\|\widehat{\Sigma} - \sigma^2 I_p\|^2 \leq 2\|\widehat{\Sigma} - \Sigma\|^2 + 2\|\Sigma - \sigma^2 I_p\|^2$$

$$\lesssim \lambda + \sqrt{\frac{(\lambda + \sigma^2)(r\lambda + p\sigma^2)}{n}} \lesssim \lambda,$$

where the first inequality can be obtained by integrating the probability bound in Lemma 6.1. Therefore, we conclude that

$$\mathbb{E}\|\widetilde{\Sigma} - \Sigma\| \lesssim \lambda\left(\sqrt{\frac{r}{n}} + \frac{\sqrt{r}(r + \log n)}{n\varepsilon} \cdot \sqrt{\log\frac{2.5}{\delta}}\right) + \sqrt{\sigma^2(\lambda + \sigma^2)}\left(\sqrt{\frac{p}{n}} + \frac{p\sqrt{(r + \log n)}}{n\varepsilon}\sqrt{\log\frac{2.5}{\delta}}\right).$$

Similar bounds can also be derived for the nuclear norm distance $\|\widetilde{\Sigma} - \Sigma\|_*$ and the general Schatten-q norm $\|\widetilde{\Sigma} - \Sigma\|_q$. The detailed proof is skipped.

6.6. *Proof of Theorem 4.2.* Some preliminary results on the KL-divergence and total variation distance between Gaussian distributions are required. Let $\mathcal{N}(\mu_1, \Sigma_1)$ and $\mathcal{N}(\mu_2, \Sigma_2)$ be two p-dimensional multivariate Gaussians, then

$$\mathrm{KL}\left(\mathcal{N}(\mu_1, \Sigma_1)\|\mathcal{N}(\mu_2, \Sigma_2)\right)$$
$$= \frac{1}{2}\left(\mathrm{Tr}\left(\Sigma_2^{-1}\Sigma_1 - I_p\right) + (\mu_2 - \mu_1)^\top \Sigma_2^{-1}(\mu_2 - \mu_1) + \log\left(\frac{\det\Sigma_2}{\det\Sigma_1}\right)\right).$$

Suppose $U_i, U_j \in \mathbb{O}_{p,r}$ satisfying $\left\|U_iU_i^\top - U_jU_j^\top\right\|_\mathrm{F} \leq \varepsilon_0$. Let $\lambda, \sigma^2 \geq 0$ be constants and define $\Sigma_i = \lambda U_iU_i^\top + \sigma^2 I_p$ and $\Sigma_j = \lambda U_jU_j^\top + \sigma^2 I_p$, respectively. Then it is easy to check that

$$\mathrm{KL}\left(\mathcal{N}(0, \Sigma_1)\|\mathcal{N}(0, \Sigma_j)\right) = \frac{1}{2}\left(\mathrm{Tr}\left(\Sigma_j^{-1}\Sigma_i - I_p\right) + \log\left(\frac{\det\Sigma_j}{\det\Sigma_i}\right)\right)$$
$$= \frac{1}{2}\left(\frac{\lambda}{\sigma^2} - \frac{\lambda}{\lambda + \sigma^2}\right)\left[r - \mathrm{Tr}\left(U_jU_j^\top U_iU_i^\top\right)\right] \leq \frac{1}{2}\frac{\lambda^2}{\sigma^2(\sigma^2 + \lambda)}\varepsilon_0^2,$$

and further by Pinsker's inequality, we have

$$\mathrm{TV}\left(\mathcal{N}(0, \Sigma_i), \mathcal{N}(0, \Sigma_j)\right) \leq \sqrt{\frac{1}{2}\mathrm{KL}\left(\mathcal{N}(0, \Sigma_i)\|\mathcal{N}(0, \Sigma_j)\right)} \leq \frac{1}{2}\varepsilon_0\sqrt{\frac{\lambda^2}{\sigma^2(\sigma^2 + \lambda)}}.$$

In order to apply Fano's lemma, we need to construct a large subset of $\mathbb{O}_{p,r}$ within which the elements are well-separated. Towards that end, we apply existing results of the packing number of Grassmannians. Indeed, by [37, Proposition 8] and [31, Lemma 5], for any $q \in [1, \infty]$, there exists an absolute constant $c' > 0$ and a subset $\mathcal{S}_q^{(p-r)} \subset \mathbb{O}_{p-r,r}$ such that for any $V_i \neq V_j \in \mathcal{S}_q^{(p-r)}$,

$$\|V_iV_i^\top - V_jV_j^\top\|_q \geq c'r^{1/q}$$

and the cardinality of $\mathcal{S}_q^{(p-r)}$ is at least $2^{r(p-r)}$. Here, $\|\cdot\|_q$ denotes the Schatten-q norm of a matrix. In particular, spectral norm is Schatten-∞ norm, Frobenius norm is Schatten-2 norm, and nuclear norm is Schatten-1 norm. Let $\varepsilon_0 > 0$ be a small number to be decided later. Now, for each $V \in \mathcal{S}_q^{(p-r)}$, we define

$$U = \begin{pmatrix} \sqrt{1 - \varepsilon_0^2}I_r \\ \sqrt{\varepsilon_0^2}V \end{pmatrix}$$

such that $U \in \mathbb{R}^{p \times r}$ and $U^\top U = I_r$. This means that, for any $V \in \mathcal{S}_q^{(p-r)}$, we can construct a $U \in \mathbb{O}_{p,r}$. This defines a subset $\mathcal{S}_q^{(p)} \subset \mathbb{O}_{p,r}$ with $\mathrm{Card}\left(\mathcal{S}_q^{(p)}\right) \geq 2^{r(p-r)}$ such that for any $U_i \neq U_j \in \mathcal{S}_q^{(p)}$,

$$\|U_iU_i^\top - U_jU_j^\top\|_q \geq \sqrt{\varepsilon_0^2(1 - \varepsilon_0^2)}\|V_i - V_j\|_q \gtrsim \sqrt{\varepsilon_0^2(1 - \varepsilon_0^2)}\|V_iV_i^\top - V_jV_j^\top\|_q \gtrsim \sqrt{\varepsilon_0^2(1 - \varepsilon_0^2)}r^{1/q}$$

and, meanwhile,

$$\|U_i U_i^\top - U_j U_j^\top\|_{\mathrm F} \lesssim \|U_i - U_j\|_{\mathrm F} \le \varepsilon_0 \|V_i - V_j\|_{\mathrm F} \le \sqrt{2r}\varepsilon_0.$$

We then consider a family of distributions as

$$\mathcal{P}\big(\mathcal{S}_q^{(p)}, \lambda, \sigma^2\big) = \{\mathcal{N}(0,\Sigma)^{\otimes n} : \Sigma = \lambda UU^\top + \sigma^2 I_p, \ U \in \mathcal{S}_q^{(p)}\} \subset \mathcal{P}(\lambda, \sigma^2),$$

whose cardinality $N := \mathrm{Card}\big(\mathcal{P}\big(\mathcal{S}_q^{(p)}, \lambda, \sigma^2\big)\big) \ge 2^{r(p-r)}$.

For $i \ne i' \in [N]$, the probability measures $P_i = \mathcal{N}(0,\Sigma_i)^{\otimes n}$ and $P_{i'} = \mathcal{N}(0,\Sigma_{i'})^{\otimes n}$ in $\mathcal{P}\big(\mathcal{S}_q^{(p)}, \lambda, \sigma^2\big)$ satisfy

$$\sum_{k \in [n]} \mathrm{TV}\left(\mathcal{N}(0,\Sigma_i), \mathcal{N}(0,\Sigma_{i'})\right) \lesssim \frac{n}{2}\sqrt{r}\varepsilon_0 \sqrt{\frac{\lambda^2}{\sigma^2(\sigma^2+\lambda)}},$$

and

$$\max_{i \ne j \in [N]} \mathrm{KL}\left(\mathcal{N}(0,\Sigma_i)^{\otimes n} \| \mathcal{N}(0,\Sigma_{i'})^{\otimes n}\right) \lesssim \frac{n}{2}\frac{\lambda^2}{\sigma^2(\sigma^2+\lambda)}\varepsilon_0^2 r.$$

To invoke Lemma 4.1, we define the metric $\rho : \mathbb{O}_{p,r} \times \mathbb{O}_{p,r} \mapsto \mathbb{R}^+$ as $\rho(U_i, U_j) := \|U_i U_i^\top - U_j U_j^\top\|_q$ for any $q \in [1,\infty]$ and take $\rho_0 \asymp \tau\varepsilon_0 r^{1/q}$,

$$l_0 = c_0 \frac{n}{2}\frac{\lambda^2}{\sigma^2(\sigma^2+\lambda)}\varepsilon_0^2 r \quad \text{and} \quad t_0 = c_0 \frac{n}{2}\sqrt{r}\varepsilon_0 \sqrt{\frac{\lambda^2}{\sigma^2(\sigma^2+\lambda)}}$$

for some small absolute constant $c_0, \tau > 0$. Then, by Lemma 4.1, for any (ε,δ)-DP estimator $\widetilde{U}$,

$$\sup_{P \in \mathcal{P}\big(\mathcal{S}_q^{(p)}, \lambda, \sigma^2\big)} \mathbb{E}\big\|\tilde{U}\tilde{U}^\top - UU^\top\big\|_q$$

$$\ge \max\left\{ \frac{\tau\varepsilon_0 r^{1/q}}{2}\left(1 - \frac{c_0 \frac{n}{2}\frac{\lambda^2}{\sigma^2(\sigma^2+\lambda)}\varepsilon_0^2 r + \log 2}{\log N}\right), \frac{\tau\varepsilon_0 r^{1/q}}{4}\left(1 \wedge \frac{N-1}{\exp\left(4\varepsilon c_0 \frac{n}{2}\sqrt{r}\varepsilon_0\sqrt{\frac{\lambda^2}{\sigma^2(\sigma^2+\lambda)}}\right)}\right)\right\},$$

where, for simplicity, we can choose $\delta = 0$. Recall that $N \ge 2^{rp/2}$ if $p \ge 2r$. We can take

$$\varepsilon_0 \asymp \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}}\sqrt{\frac{p}{n}} + \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}}\frac{p\sqrt{r}}{n\varepsilon},$$

and get

$$\sup_{P \in \mathcal{P}\big(\mathcal{S}_q^{(p)}, \lambda, \sigma^2\big)} \mathbb{E}\big\|\tilde{U}\tilde{U}^\top - UU^\top\big\|_q \gtrsim \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}}\cdot r^{1/q}\sqrt{\frac{p}{n}} + \sqrt{\frac{\sigma^2(\lambda+\sigma^2)}{\lambda^2}}\cdot r^{\frac{1}{2}+\frac{1}{q}}\frac{p}{n\varepsilon}.$$

Since a trivial upper of $\|\widetilde{U}\widetilde{U}^\top - UU^\top\|_q \le (2r)^{1/q}$ and $\mathcal{P}\big(\mathcal{S}_q^{(p)}, \lambda, \sigma^2\big) \subset \mathcal{P}(\lambda, \sigma^2)$, we conclude that, for any $q \in [1,\infty]$,

$$\inf_{\tilde{U}} \sup_{P \in \mathcal{P}(\lambda, \sigma^2)} \mathbb{E}\big\|\tilde{U}\tilde{U}^\top - UU^\top\big\|_q \gtrsim \left(\frac{\sigma^2}{\lambda} + \sqrt{\frac{\sigma^2}{\lambda}}\right)\left(r^{1/q}\sqrt{\frac{p}{n}} + r^{\frac{1}{2}+\frac{1}{q}}\frac{p}{n\varepsilon}\right)\bigwedge r^{1/q},$$

where the infimum is taken over all possible (ε,δ)-DP algorithms. Now it suffices to choose $q = 1, 2, \infty$ to obtain the bounds in nuclear norm, Frobenius norm, and spectral norm, respectively.

6.7. *Proof of Theorem 4.3.* Note that the two terms in the minimax lower bounds are contributed by estimating the eigenvalues and eigenvectors, separately. We begin with the term related to estimating eigenvectors. Consider a subset $\mathcal{P}_1(\lambda, \sigma^2) \subset \mathcal{P}(\lambda, \sigma^2)$ defined by

$$\mathcal{P}_1(\lambda, \sigma^2) := \left\{ \mathcal{N}(0, \Sigma) : \Sigma = \lambda UU^\top + \sigma^2 I_p \quad \text{and} \quad U \in \mathbb{O}_{p,r} \right\},$$

where we assume λ and σ^2 are both known. If λ is already known, it suffices to estimate $UU^\top$ differentially privately by an estimator $\widetilde{U}\widetilde{U}^\top$ so that we can construct the covariance matrix estimator $\widetilde{\Sigma} = \lambda \widetilde{U}\widetilde{U}^\top + \sigma^2 I_p$. Therefore,

$$
\inf_{\widetilde{\Sigma}} \sup_{P \in \mathcal{P}_1(\lambda, \sigma^2)} \mathbb{E} \big\| \widetilde{\Sigma} - \Sigma \big\| \geq \inf_{\widetilde{U}} \sup_{P \in \mathcal{P}_1(\lambda, \sigma^2)} \lambda \cdot \mathbb{E} \big\| \widetilde{U}\widetilde{U}^\top - UU^\top \big\|
$$

(20)

$$
\gtrsim \sqrt{\sigma^2(\lambda + \sigma^2)} \left(\sqrt{\frac{p}{n}} + \frac{\sqrt{r}p}{n\varepsilon} \right) \bigwedge \lambda,
$$

and

$$
\inf_{\widetilde{\Sigma}} \sup_{P \in \mathcal{P}_1(\lambda, \sigma^2)} \mathbb{E} \big\| \widetilde{\Sigma} - \Sigma \big\|_{\mathrm{F}} \geq \inf_{\widetilde{U}} \sup_{P \in \mathcal{P}_1(\lambda, \sigma^2)} \lambda \cdot \mathbb{E} \big\| \widetilde{U}\widetilde{U}^\top - UU^\top \big\|_{\mathrm{F}}
$$

(21)

$$
\gtrsim \sqrt{\sigma^2(\lambda + \sigma^2)} \left(\sqrt{\frac{pr}{n}} + \frac{rp}{n\varepsilon} \right) \bigwedge \lambda\sqrt{r},
$$

where the last inequalities in (20) and (21) are both due to Theorem 4.2. These establish the second term in the minimax lower bounds of Theorem 4.3. More generally, we can establish the minimax lower bound in Schatten-q norms:

$$
\inf_{\widetilde{\Sigma}} \sup_{P \in \mathcal{P}_1(\lambda, \sigma^2)} \mathbb{E} \big\| \widetilde{\Sigma} - \Sigma \big\|_{\mathrm{q}} \geq \inf_{\widetilde{U}} \sup_{P \in \mathcal{P}_1(\lambda, \sigma^2)} \lambda \cdot \mathbb{E} \big\| \widetilde{U}\widetilde{U}^\top - UU^\top \big\|_q
$$

(22)

$$
\gtrsim \sqrt{\sigma^2(\lambda + \sigma^2)} \left(r^{1/q}\sqrt{\frac{p}{n}} + \frac{pr^{\frac{1}{2}+\frac{1}{q}}}{n\varepsilon} \right) \bigwedge \lambda r^{1/q},
$$

for any $q \in [1, \infty]$.

We now establish the first term in the minimax lower bounds $\lambda\big(\sqrt{r/n} + r/(n\varepsilon)\big)$, which is contributed by estimating the singular values. It is unrelated to the nuisance variance σ^2 and the eigenvectors U. Without loss of generality, we can assume $\sigma^2 = 0$ and U_V in the format

$$
U_V = \begin{pmatrix} V \\ \mathbf{0}_{(p-r)\times r} \end{pmatrix},
$$

where $V = [V_0, V_{0\perp}] \in \mathbb{O}_{r,r}$ with some $V_0 \in \mathbb{O}_{r,r/4}$. It is easy to check that $U_V^\top U_V = V^\top V = I_r$. Define

$$
\Lambda_0 = \mathrm{diag}(\underbrace{2\lambda, \cdots, 2\lambda}_{r/4}, \lambda, \cdots, \lambda),
$$

which is an $r \times r$ diagonal matrix. For any $V = [V_0, V_{0\perp}] \in \mathbb{O}_{r,r}$ with $V_0 \in \mathbb{O}_{r,r/4}$, we consider the following covariance matrix

$$
\Sigma_{V_0} := U_V \Lambda_0 U_V^\top = \begin{pmatrix} \lambda I_r + \lambda V_0 V_0^\top & \mathbf{0}_{r\times(p-r)} \\ \mathbf{0}_{(p-r)\times r} & \mathbf{0}_{(p-r)\times(p-r),} \end{pmatrix}
$$

To this end, we define a subset $\mathcal{P}_2(\lambda) \subset \mathcal{P}(\lambda, \sigma^2)$ as

$$
\mathcal{P}_2(\lambda) := \left\{ \mathcal{N}(0, \Sigma_{V_0}) : \Sigma_{V_0} = \begin{pmatrix} \lambda I_r + \lambda V_0 V_0^\top & \mathbf{0}_{r\times(p-r)} \\ \mathbf{0}_{(p-r)\times r} & \mathbf{0}_{(p-r)\times(p-r)} \end{pmatrix}, V_0 \in \mathbb{O}_{r,r/4} \right\}.
$$

We are interested in the minimax lower bound:

$$\inf_{\widetilde{\Sigma}} \sup_{P \in \mathcal{P}_2(\lambda)} \mathbb{E}\|\widetilde{\Sigma} - \Sigma\| \quad \text{and} \quad \inf_{\widetilde{\Sigma}} \sup_{P \in \mathcal{P}_2(\lambda)} \mathbb{E}\|\widetilde{\Sigma} - \Sigma\|_{\mathrm{F}},$$

where the infimum is taken over all the possible (ε, δ)-DP algorithms.

Observe that if a random vector $X = (X_1, \cdots, X_p)^\top \sim \mathcal{N}(0, \Sigma_{V_0})$, it means that only the first r entries of X are random variables since other variables are simply zeros with probability one. This suggests that it suffices to consider the reduced problem of estimating an $r \times r$ spiked covariance matrix. Towards that end, define a family of distributions of r-dimensional random vector

$$\mathcal{P}_2^{(0)}(\lambda) := \Big\{ \mathcal{N}(0, \Sigma_{0V_0}) : \ \Sigma_{0V_0} = \lambda I_r + \lambda V_0 V_0^\top, \ V_0 \in \mathbb{O}_{r, r/4} \Big\}.$$

Given i.i.d. observations $X_1^{(0)}, \cdots, X_n^{(0)} \sim P \in \mathcal{P}_2^{(r)}(\lambda)$, we aim to estimate the covariance matrix Σ_0 with an (ε, δ)-differentially private algorithm. Clearly, by definition,

$$(23) \qquad \inf_{\widetilde{\Sigma}} \sup_{P \in \mathcal{P}_2(\lambda)} \mathbb{E}\|\widetilde{\Sigma} - \Sigma\|_q = \inf_{\widetilde{\Sigma}_0} \sup_{P \in \mathcal{P}_2^{(0)}(\lambda)} \mathbb{E}\|\widetilde{\Sigma}_0 - \Sigma_0\|_q,$$

for any Schatten-q norms. It therefore suffices to study the RHS of (23), which is the differentially private minimax lower bound for estimating an $r \times r$ spiked covariance matrix with rank $r/4$. Without loss of generality, we can still assume λ is known and we can immediately invoke the bounds (20), (21), (22) by replacing $\sigma^2 = \lambda$ and $r \leftarrow r/4$, $p \leftarrow r$ there, and conclude that

$$(24) \qquad \inf_{\widetilde{\Sigma}_0} \sup_{P \in \mathcal{P}_2^{(0)}(\lambda)} \mathbb{E}\|\tilde{\Sigma} - \Sigma\| \gtrsim \lambda\Big(\sqrt{\frac{r}{n}} + \frac{r^{3/2}}{n\varepsilon}\Big) \bigwedge \lambda,$$

and

$$(25) \qquad \inf_{\widetilde{\Sigma}_0} \sup_{P \in \mathcal{P}_2^{(0)}(\lambda)} \mathbb{E}\|\tilde{\Sigma} - \Sigma\|_{\mathrm{F}} \gtrsim \lambda\Big(\frac{r}{\sqrt{n}} + \frac{r^2}{n\varepsilon}\Big) \bigwedge \lambda\sqrt{r},$$

or more generally,

$$(26) \qquad \inf_{\widetilde{\Sigma}_0} \sup_{P \in \mathcal{P}_2^{(0)}(\lambda)} \mathbb{E}\|\tilde{\Sigma} - \Sigma\|_q \gtrsim \lambda\Big(\frac{r^{\frac{1}{2}+\frac{1}{q}}}{\sqrt{n}} + \frac{r^{\frac{3}{2}+\frac{1}{q}}}{n\varepsilon}\Big) \bigwedge \lambda r^{1/q},$$

for any $q \in [1, +\infty]$.

Finally, putting together (20)-(22) and (24)-(26), we conclude that

$$\inf_{\widetilde{\Sigma}} \sup_{P \in \mathcal{P}(\lambda, \sigma^2)} \mathbb{E}\|\widetilde{\Sigma} - \Sigma\|_q \gtrsim \left(\lambda\Big(\frac{r^{\frac{1}{2}+\frac{1}{q}}}{\sqrt{n}} + \frac{r^{\frac{3}{2}+\frac{1}{q}}}{n\varepsilon}\Big) + \sqrt{\sigma^2(\lambda+\sigma^2)}\Big(r^{1/q}\sqrt{\frac{p}{n}} + \frac{pr^{\frac{1}{2}+\frac{1}{q}}}{n\varepsilon}\Big) \right) \bigwedge \lambda r^{1/q}.$$

Now by setting $q = 1, 2, \infty$, we complete the proof.

SUPPLEMENTARY MATERIAL

Supplement to "Optimal Differentially Private PCA and Estimation for Spiked Covariance Matrices"

In this supplement, we present the proofs of Lemmas 2.2, 3.1, 4.1, and 6.5, as well as Corollary 5.2. Additionally, we provide the proofs for technical Lemmas 6.2, 6.2, and 6.4.

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