Calculation of Unconditional Variance for the AR(2) Model

The AR(2) model for y_t is

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t,$$

where ϵ_i are i.i.d with variance σ^2. Assume that ϕ_1 and ϕ_2 are such that $\{y_t\}$ is stationary.

Denote $\text{Cov}(y_t, y_{t-k})$ as $\gamma(k)$. Since $E[y_t] = 0$, $\gamma(k) = E[y_t y_{t-k}]$, and so $\text{Var}(y_t) = \gamma(0) = E(y_t^2)$. Note that for $j > k$, $E(\epsilon_j y_k) = E(\epsilon_j)E(y_k) = 0$. With this in mind, $\text{Var}(y_t)$ can be written as follows:

$$\begin{aligned}
\text{Var}(y_t) = \gamma(0) &= E((\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t)(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t)) \\
&= E(\phi_1^2 y_{t-1}^2 + 2\phi_1\phi_2 y_{t-1} y_{t-2} + \phi_2^2 y_{t-2}^2 + \epsilon_t^2) \\
&= \phi_1^2 \gamma(0) + 2\phi_1\phi_2\gamma(1) + \phi_2^2\gamma(0) + \sigma^2 \\
&= \gamma(0)(\phi_1^2 + \phi_2^2) + 2\phi_1\phi_2\gamma(1) + \sigma^2 \quad (\star)
\end{aligned}$$

It is now apparent that to find $\gamma(0)$, we must first calculate $\gamma(1)$. Note that since the process is stationary with zero mean,

$$\gamma(1) = E(y_t y_{t-1}) = E(y_t y_{t+1}).$$

This implies that we can express $\gamma(1)$ as follows:

$$\begin{aligned}
\gamma(1) &= E(y_t y_{t+1}) \\
&= E(y_t(\phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1})) \\
&= \phi_1\gamma(0) + \phi_2\gamma(1) + E(y_t\epsilon_{t+1}) \\
&= \phi_1\gamma(0) + \phi_2\gamma(1)
\end{aligned}$$

Rearranging terms, we see $\gamma(1) = \frac{\phi_1\gamma(0)}{1-\phi_2}$. Substituting this expression into $(\star)$, we find:

$$\begin{aligned}
\gamma(0) &= \gamma(0)(\phi_1^2 + \phi_2^2) + 2\phi_1\phi_2\frac{\phi_1\gamma(0)}{1-\phi_2} + \sigma^2 \\
\gamma(0)(1 - \phi_1^2 - \phi_2^2) &= \frac{2\phi_1^2\phi_2\gamma(0)}{1-\phi_2} + \sigma^2 \\
\gamma(0)\left(1 - \phi_1^2 - \phi_2^2 - \frac{2\phi_1^2\phi_2}{1-\phi_2}\right) &= \sigma^2 \\
\gamma(0) &= \frac{\sigma^2}{1 - \phi_1^2 - \phi_2^2 - \frac{2\phi_1^2\phi_2}{1-\phi_2}}
\end{aligned}$$

$$\text{This implies } \text{Var}(y_t) = \frac{\sigma^2}{1 - \phi_1^2 - \phi_2^2 - \frac{2\phi_1^2\phi_2}{1-\phi_2}}.$$