The 1-Dimensional Discrete Voronoi Game*

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Abstract

In this paper we study the discrete version of the 1-dimensional (continuous) Voronoi game introduced by Ahn et al. [1]. The discrete Voronoi game in dimension 1, consists of two competing players P1 and P2 and a set of N users placed on a line-segment. The players alternately place one facility each on the line-segment for R-rounds, where the objective is to maximize their own total payoffs. We prove bounds on the worst-case (over the arrangement of the N users) payoffs of the two players, and show that they are often tight. We also compute the complexities of the optimal payoff functions and discuss algorithms for finding the optimal strategies of the players, in the 2-round game.

Keywords: Computational geometry, Competitive facility location, Game theory, Voronoi diagram

1 Introduction

Competitive facility location is concerned with the favorable placement of facilities by competing market players [13, 14]. It goes back to the 1929 seminal paper by Hotelling [16] which introduced the competitive facility location problem when the users were placed uniformly on a line segment (see also Eaton and Lipsey [12]). Facilities and users are generally modeled as points in a pre-specified arena (generally a subset of $\mathbb{R}^1/\mathbb{R}^2$). The set of users (demands) is a subset of the arena, which can be either discrete, consisting of finitely many points, or continuous, that is, a region where every point is considered to be a user. We assume that the facilities are equally equipped in all respects, and a user always avails the service from its nearest facility. Consequently, each facility has its *service zone*, consisting of the set of users that are served by it, and the goal is to find placement of facilities which maximize the cardinality or the area of its service zone, depending on whether the demand region is discrete or continuous, respectively. For a recent survey on the applications of competitive facility location in economics and operations research, refer to [11].

Ahn et al. [1] introduced a game-theoretic analogue of such problems for 1-dimensional arenas (line-segment/circle) with continuous demands. The game consists of the 2 players P1 and P2

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alternately placing disjoint set of facilities in the arena. In this case, the payoff of player P1/P2 is the area of the region that is closer to the facilities owned by P1/P2 than to the other player, and the player which finally owns the larger area is the winner of the game. They showed that when the players place one facility each for R-rounds, the second player always has a winning strategy that guarantees a payoff of $1/2 + \varepsilon$, with $\varepsilon > 0$. However, the first player can force ε to be arbitrarily small.

In this paper, we study the analogous version of the game for discrete demand regions: Given a positive integer $R \geq 1$, the 1-*dimensional R-round discrete Voronoi game* consists of two players P1 and P2 and a set U of N users on a line-segment $L \subset \mathbb{R}$. The players alternately place one facility each for R-rounds, with the objective to maximize their own total payoffs, where the payoff of P1/P2 is the cardinality of the set of points in U which are closer to a facility owned by P1/P2 than to every facility owned by P2/P1. To define this more formally, we introduce some notation: Given a set $\mathcal{F} \subset \mathbb{R}$ of facilities, define for every $f \in \mathcal{F}$,

$$U(f, \mathcal{F}) = \{u_a \in U : |u_a - f| < |u_a - h|, \text{ for all } h \in \mathcal{F} \backslash \{f\}\}, \tag{1.1}$$

the set of users which are closest to f. Then, for any placement of facilities S_1 by P1 and S_2 by P2, the payoff of P2 is $\mathcal{P}_2(S_1, S_2|U) = |\bigcup_{f \in S_2} U(f, S_1 \bigcup S_2)|$. Similarly, the payoff of P1, $\mathcal{P}_1(S_1, S_2|U) = |U| - \mathcal{P}_2(S_1, S_2|U)$. Note that this definition implies that if an user is equidistant from a facility in S_1 and another facility in S_2, then it contributes to the payoff of P1, that is, ties are broken in favor of P1. We will assume that facilities are not allowed to overlap with themselves or with the set of users at any stage of the game. Given a set of users U, denote by $\eta_2(R|U)$ the maximum possible payoff P2 can attain against any adversarial strategy of P1, when the game is played for R-rounds. The optimal payoff of P1 is defined similarly and will be denoted by $\eta_1(R|U)$.

Theorem 1.1. *For any $R \geq 2$, the following hold:*

(a) For any placement U of N users in the segment L, $\eta_2(R|U) \geq N \left(\frac{\sqrt{8(R+1)}-1}{2R} \right) - (R-1)$.

(b) For any placement U of N users in the segment L, $\eta_1(R|U) \geq \lfloor \frac{N}{R+1} \rfloor$. Moreover, there is a placement U_0 of N users such that $\eta_1(R|U_0) \leq \lceil \frac{N}{R+1} \rceil + R$.

The proof of the above theorem is given in Section 2. The lower bound on $\eta_1(R|U)$ follows by a trivial pigeon-hole argument. However, quite surprisingly, it turns out there is an arrangement of users for which this bound is attained (up to constant factors). In fact, the theorem gives the exact asymptotics of the *normalized worst-case payoff* of P1, that is,

$$\lim_{N \to \infty} \frac{\inf_{U:|U|=N} \eta_1(R|U)}{N} = \frac{1}{R+1},$$

where the infimum is taken over all arrangement of N users on L. This shows how the arrangement of users affect the payoff of P1, and is in sharp contrast with the continuous Voronoi game of Ahn et al. [1], where P1 can always have a payoff which is arbitrarily close to 1/2.

Obtaining lower bounds on the payoff of P2 is more challenging. The bound in the theorem above, shows that the normalized worst-case payoff of P2 is at least $\frac{1}{2R}(\sqrt{8(R+1)}-1) = \Theta(1/\sqrt{R})$. In the special case $R = 2$, this can be improved as follows:

Corollary 1.2. *For any placement U of N users in the segment L, $\eta_2(2|U) \geq \lfloor N/2 \rfloor$. Moreover, there is a placement U_0 of N users such that $\eta_2(2|U_0) \leq \lceil N/2 \rceil$.*

The above corollary shows that, irrespective of the arrangement of the users U on L, P2 can only lose the 2-round game by at most a single user. This also implies that $\lim_{N\to\infty} \frac{1}{N} \inf_{U:|U|=N} \eta_2(R|U) = \frac{1}{2}$, that is, P2 can always asymptotically tie the game, irrespective of the placement of the users on L. Note that in the 1-round game, P2 can trivially tie the game, with a payoff of at least $\lceil N/2 \rceil$ by placing its facility immediately next to the facility of P1, either to the right or to the left, depending on which side has more users in U. Whether P2 can always asymptotically tie the game, for any $R \geq 3$, remains open.

Remark 1.1. The above results show how the arrangement of the users can affect the payoff of the players, for example, the normalized worst case pay-off of P1 can be as bad as $\frac{1}{R+1}$, and as good as $\frac{1}{2}$ (when the users are equi-distributed on L). This does not arise in the continuous game of Ahn et al. [1], because the uniformity of the demands allows P1 to always get payoff arbitrarily close to $\frac{1}{2}$. Therefore, for the discrete Voronoi game, this raises the following algorithmic question: given a placement of the users, how efficiently can one find the optimal strategy of the players. In general, this appears to be a difficult problem with algorithmic complexity increasing exponentially with the number of rounds.

In the theorem below, we consider the 2-round game, and provide algorithms for computing the optimal strategies of both the players.

Theorem 1.3. *For the 2-round game the following holds:*

(a) The optimal strategies of P2 and P1 in round 2 can be computed in $O(N)$ and $O(N^2)$ times, respectively.

(b) The optimal strategies of P2 and P1 in round 1 can be computed in $O(N^5)$ and $O(N^9)$ times, respectively.

This theorem is proved in Section 3, which involves computing how the payoffs of the players change, assuming each player plays optimally, that is, to maximize their own eventually payoffs, in the subsequent rounds.

1.1 Related Work

Dehne et al. [10] studied a competitive facility location problem for continuous demand regions in $\mathbb{R}^2$, where the problem is to find a new point q amidst a set of n existing points $\mathcal{F}$ such that the Voronoi region of q is maximized. They showed that when the points in $\mathcal{F}$ are in convex position, the area function has only a single local maximum inside the region where the set of Voronoi neighbors do not change. For the same problem, Cheong et al. [8] gave a near-linear time algorithm that determines the location of the new optimal point approximately, when the points in $\mathcal{F}$ are in general position. In the discrete user case, the analogous problem is to place a set of new facilities amidst a set of existing ones such that the number of users served by the new facilities is maximized [6, 7].

Cheong et al. [9] studied the 1-round (continuous) Voronoi game in $\mathbb{R}^2$ for a square-shaped demand region, which was later extended by Fekete and Meijer [15] to rectangular demand regions.

Here, P1 followed by P2, places m facilities in the demand region, and the player with the larger Voronoi area wins the game. Recently, variants of these games when the demand region is a graph equipped with the shortest-path distance [2] has been studied.

Banik et al. [3] studied the one-round discrete Voronoi game in $\mathbb{R}$, where, given a set U of N users on a line, P1 chooses a set of m facilities, following which P2 chooses another disjoint set of m facilities, and the objective of both the players is to maximize the number of users they serve. The authors showed that if the sorted order of the points in U along the line is known, then the optimal strategy of P2, given any placement of facilities by P1, can be computed in $O(N)$ time, and the optimal strategy of P1 can be computed in $O(N^{m-\lambda_m})$ time, where $0 < \lambda_m < 1$, is a constant depending only on m. In 2-dimensions, exact and approximation algorithms for the discrete Voronoi game in some special cases, were recently obtained in [4] and [5], respectively.

2 Proofs

Throughout this section we will assume that the playing arena is the interval $L = [A, B]$. Moreover for any two points $a, b \in L$, with $a < b$, we will denote by $U[a, b] = |[a, b] \cap U|$, the number of users in U in the interval $[a, b]$. The definition is naturally modified when one or both of the endpoints of the interval are open.

2.1 Proof of Theorem 1.1

Throughout the paper, we will use the phrase 'point b is placed immediately to the right/left of another point a' to mean that $b = a \pm \varepsilon$, where $\varepsilon > 0$ is chosen to be arbitrarily small. In fact, choosing any $\varepsilon < \frac{1}{1000}\min_{0\leq j\leq N}|u_{j+1} - u_j|$, where $u_0 = A$ and $u_N = B$, would suffice for our purpose. We begin the following simple observation:

Observation 2.1. *Given any placement $\mathcal{F}$ of M facilities in an interval $L_0 = [A_0, B_0] \subseteq L$, there exists $s \in L_0$ such that at least $\frac{1}{2M}|U \cap L_0|$ users in $U \cap L_0$ are closer to s than to the M users in $\mathcal{F}$.*

Proof. Let $\mathcal{F} = \{f_1, f_2, \cdots, f_M\}$ such that $f_1 < f_2 < \cdots < f_M$. If either $U[A_0, f_1] \geq \frac{1}{2M}|U \cap L_0|$ or $U[f_M, B_0] \geq \frac{1}{2M}|U \cap L_0|$, then by placing s immediately to the left of f_1 or right of f_M, respectively, the result follows.

Therefore, assume that $U[A_0, f_1] < \frac{1}{2M}|U \cap L_0|$ and $U[f_M, B_0] < \frac{1}{2M}|U \cap L_0|$. Then, by the pigeonhole principle, there exists $1 \leq j \leq M - 1$ such that $U[f_j, f_{j+1}] \geq \frac{1}{M}|U \cap L_0|$. Therefore, placing s immediately to the right of f_j or left f_{j+1} it is possible for s to get to at least $\frac{1}{2M}|U \cap L_0|$ users in $U \cap L_0$. □

Now, we propose a strategy for P2 which gives the required lower bound in Theorem 1.1. In the first $R-1$ rounds, P2 places its facilities at $s_1, s_2, \ldots, s_{R-1}$, respectively, where $s_1, s_2, \ldots, s_{R-1}$ are chosen such that $U[A, s_1] = \lfloor \frac{N}{2R} \rfloor$ and $U[s_j, s_{j+1}] = \lfloor \frac{N}{2R} \rfloor$, for $1 \leq j \leq R - 2$. Suppose, after all the R rounds there are $1 \leq b \leq R$ facilities of P1 in the interval $[A, s_{R-1}]$ and $R - b$ in the interval $[s_{R-1}, B]$. If $b = R$, then all points in the interval $[s_{R-1}, B]$ belong to P2, and $\eta_2(R|U) \geq N - \lfloor \frac{N}{2R} \rfloor (R - 1) \geq \frac{R+1}{2R}N$. Otherwise, $1 \leq b \leq R - 1$ and by Observation 2.1 P2 can chose $s_R \in [s_{R-1}, B]$ such that P2 gets at least $\frac{1}{2(R-b)}U[s_{R-1}, B] = \frac{N-\lfloor \frac{N}{2R} \rfloor(R-1)}{2(R-b)}$ users. Moreover,

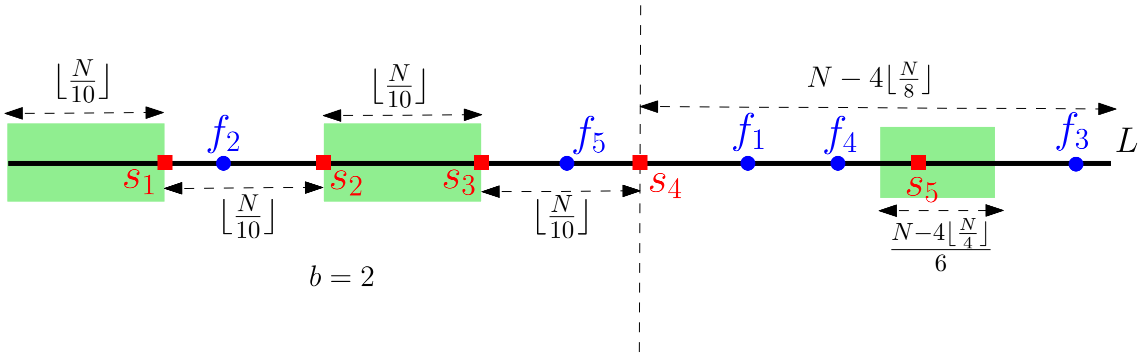

Figure 1: Proof of Theorem 1.1: Strategy of P2 in the $R = 5$ round game, when $b = 2$ (where b is the number facilities of P1 to the left of s_4).

since at least $R - b - 1$ of the intervals $[A, s_1], [s_1, s_2], \ldots, [s_{R-2}, s_{R-1}]$ contains no point from P1, the total payoff P2 is at least

$$\begin{aligned}
\eta_2(R|U) &\geq (R-b-1)\left\lfloor \frac{N}{2R} \right\rfloor + \frac{N - \lfloor \frac{N}{2R} \rfloor (R-1)}{2(R-b)} \\
&\geq (R-b-1)\left(\frac{N}{2R}\right) + \frac{N - \frac{N}{2R}(R-1)}{2(R-b)} - (R-b-1) \qquad \text{(using } x - 1 \leq \lfloor x \rfloor \leq x\text{)} \\
&\geq \frac{N}{2R}\left(R - b + \frac{2(R+1)}{R-b}\right) - \frac{N}{2R} - (R-1) \\
&\geq N\left(\frac{\sqrt{8(R+1)} - 1}{2R}\right) - (R-1),
\end{aligned}$$

where the last step uses the fact that the function $f(x) := x + \frac{2(R+1)}{x}$ is minimized at $x = \sqrt{2(R+1)}$. This completes the proof of part (a).

Next, we prove part (b). The strategy for P1 which gives the required lower bound on the payoff of P1 is as follows: In the R-rounds P1 places its facilities at $f_1, f_2, \ldots, f_R$, respectively, such that $U[A, f_1] \geq \lfloor \frac{N}{R+1} \rfloor$, $U[f_R, B] \geq \lfloor \frac{N}{R+1} \rfloor$ and $U[f_j, f_{j+1}] \geq \lfloor \frac{N}{R+1} \rfloor$, for $1 \leq j \leq R - 1$. By the pigeonhole principle, for any placement of R facilities by P2, at least one of these $R + 1$ intervals $[A, f_1], [f_1, f_2], \ldots, [f_{R-1}, f_R], [f_R, B]$ contains no points of P2, which implies $\eta_1(R|U) \geq \lfloor \frac{N}{R+1} \rfloor$.

To show this bound is attained, let $U_0 = \{2, 4, \ldots, 2^N\} \subset L_0 := [1, 2^{N+1}]$. (Note that the points can be scaled to lie in any pre-specified line segment L, if required.)

Observation 2.2. *Let U_0 be as above and $1 \leq K, K' \leq N$. Suppose $f \in (2^K, 2^{K+1})$ and $f' \in (2^{K+K'}, 2^{K+K'+1})$ be two facilities of P1 such that the interval (f, f') contains no other facilities of P1 and P2. Then P2 can place a facility s immediately to the right of f and serve $K' - 1$ users in $U[f, f']$.*

Proof. Note that $\frac{1}{2}(s + f') > 2^{K-1} + \frac{f'}{2} \geq 2^{K-1} + 2^{K+K'-1} > 2^{K+K'-1}$. This implies, s serves $K' - 1$ users in the interval (f, f'). □

This observation can be used to construct a strategy for P2. Depending on the placement f_1 of P1 in round 1, there are two cases:

– $U_0[1, f_1] \leq \lceil \frac{N}{R+1} \rceil$: Then in round 1 P2 places s_1 immediately to the right of f_1. More generally, in the r-th round, where $1 \leq r \leq R$, given the placement $f_1, f_2, \ldots, f_r$ by P1, P2 places s_r immediately to the right of f_r. Let $f'_1 < f'_2 < \ldots < f'_R$ be the sorted order of the facilities in P1 after R rounds are completed. This decomposes the segment L_0 into $R+1$ intervals $[f'_0, f'_1], [f'_1, f'_2], [f'_2, f'_3], \cdots, [f'_{R-1}, f'_R], [f'_R, f'_{R+1}]$, where $f'_0 = 1$ and $f'_{R+1} = 2^N$. Now, by Observation 2.2, for every $1 \leq r \leq R$, s'_r serves $U_0[f'_r, f'_{r+1}] - 1$, which implies that the payoff of P2 is at least

$$\sum_{r=1}^{R}(U_0[f'_r, f'_{r+1}] - 1) = \sum_{r=0}^{R} U_0[f'_r, f'_{r+1}] - U_0[f'_0, f'_1] - R \geq N - \left\lceil \frac{N}{R+1} \right\rceil - R.$$

Therefore, the payoff of P1 is at most $\lceil \frac{N}{R+1} \rceil + R$.

– $U[1, f_1] > \lceil \frac{N}{R+1} \rceil$: In this case, in round 1 P2 places s_1 such that $U[1, s_1] = \lceil \frac{N}{R+1} \rceil$. Define $S_0 := \{2 \leq j \leq R : f_j \in [1, s_1]\}$, that is, the rounds in which P1 places its facilities to the left of P1. Again there are two cases:

- $|S_0| \neq 0$: Let $j_1 < j_2 < \ldots < j_{|S_0|}$ be the elements in S_0 in increasing order. Then starting round 2 the strategy of P2 is as follows: If $j \notin S_0$, then place s_j immediately to right of f_j. On the other hand, for $j_b \in S_0$, place s_{j_b} immediate to right of f_1, but left s_{b-1}, for $1 \leq b \leq |S_0|$ (setting $s_0 := 2^N$). Then by Observation 2.2, it follows that the payoff of P1 is at most $\lceil \frac{N}{R+1} \rceil + R - |S_0| \leq \lceil \frac{N}{R+1} \rceil + R$ (since, at best P1 can get all the users in $[1, s_1]$ and one user each for the facilities in $\{f_j : j \notin S_0\}$).
- $|S_0| = 0$: Let $1 \leq r \leq R$, and $f_1, s_1, f_2, s_2, \ldots, f_r, s_r$ be the placements of the players P1 and P2 in the first r rounds (recall that s_1 is chosen such that $U[1, s_1] = \lceil \frac{N}{R+1} \rceil$). Assume that $f_1^{(r)} < f_2^{(r)} < \ldots < f_r^{(r)} < f_{r+1}^{(r)} := B$ be the sorted order of the points on L. Now, suppose in the $(r+1)$-th round, P1 places at f_{r+1} such that $f_{r+1} \in [f_j^{(r)}, f_{j+1}^{(r)}]$, for some $1 \leq j \leq r$. Again there are two cases:
  – $U[f_j^{(r)}, f_{j+1}^{(r)}] \geq \lceil \frac{N}{R+1} \rceil$: Then P2 places s_r right next to f_r.
  – $U[f_j^{(r)}, f_{j+1}^{(r)}] < \lceil \frac{N}{R+1} \rceil$. Now, if r is the first time this happens then P2 places s_r immediately to the right of f_1. For every subsequently round when this happens, P2 places s_r immediately to the right of f_r.

  To see that this strategy works, let $f_1^{(R)} < f_2^{(R)} < \ldots < f_R^{(R)} < f_{R+1}^{(R)} := B$ be the sorted order of the points after the R rounds. Note that $\sum_{i=1}^{R} U[f_j^{(R)}, f_j^{(R+1)}] \leq N - \lceil \frac{N}{R+1} \rceil$, which implies that there exists at least one $1 \leq j \leq R$ such that

$$U[f_j^{(R)}, f_j^{(R)}] < \left\lceil \frac{N - \lceil \frac{N}{R+1} \rceil}{R} \right\rceil \leq \left\lceil \frac{N}{R+1} \right\rceil.$$

  Therefore, the strategy of P2 described above implies that there exists some $s_i^{(R)}$ immediately to right of $f_i^{(R)}$, for all $i \in \{1, 2, \ldots, R\} \backslash \{j_0\}$, where j_0 is such that $U[f_{j_0}^{(R)}, f_{j_0}^{(R)}] < \lceil \frac{N}{R+1} \rceil$. This implies that the total payoff of P1 is at most $\lceil \frac{N}{R+1} \rceil + R$.

2.2 Proof of Corollary 1.2

We begin by showing that there exists a strategy of P2 with payoff at least $\lceil N/2 \rceil$. To begin with note that for any placement f_1 by P1 either $U[A, f_1] \leq \lfloor N/2 \rfloor$ or $U[f_1, B] \leq \lfloor N/2 \rfloor$. Without loss of generality, we assume that, $U[A, f_1] := K \leq \lfloor N/2 \rfloor$. Then, in round 1 P2 places $s_1 \in L$ such that $U[f_1, s_1] = \lceil N/2 \rceil$ (see Figure 2(a)). Now, consider the following 3 cases:

- $f_2 \in [A, f_1)$: In this case, placing s_2 immediately to the right of f_1, gives P2 a payoff of $N - K \geq \lceil N/2 \rceil$ (see Figure 2(a)).

- $f_2 \in (f_1, s_1)$: Then placing s_2 immediately to the left of f_1 gives P2 a payoff of $\lfloor N/2 \rfloor$ (see Figure 2(b)).

- $f_2 \in (s_1, B]$: In this case, again placing s_2 immediately to the right of f_1, gives P2 a payoff of $\lceil N/2 \rceil$ (see Figure 2(c)).

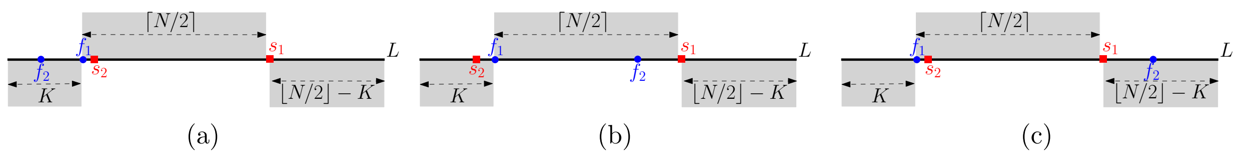

Figure 2: The different cases in the proof of Corollary 1.2.

The argument above shows that $\eta_2(2|U) \geq \lfloor N/2 \rfloor$. To show that this is attained, let U_0 be N equally points in L, that is, $U_0 = \{A + j\frac{B-A}{N+1} : 1 \leq j \leq N\}$. In the first round P1 places at f_1, such that $U[A, f_1] = \lfloor N/4 \rfloor$. Now, depending on the placement s_1 of P1 in round 1, there are three cases:

- $s_1 < f_1$: In round 2, P2 places at $f_2 \in (s_1, B]$ such that $U[f_2, B] = \lfloor N/4 \rfloor$. Then, it is easy to see that for any placement s_2 of P2 in round 2, the payoff of P2 is at most $\lfloor N/4 \rfloor + \lceil \frac{N - 2\lfloor N/4 \rfloor}{2} \rceil = \lceil N/2 \rceil$.

- $s_1 > f_1$ and $U[f_1, s_1] \leq \lceil N/2 \rceil$: As in the previous case, P2 places at $f_2 \in (s_1, B]$ such that $U[f_2, B] = \lfloor N/4 \rfloor$. Again, this guarantees that the payoff of P2 is at most $\lceil N/2 \rceil$.

- $U[f_1, s_1] > \lceil N/2 \rceil$: In round 2, P2 places at $f_2 \in (f_1, s_1)$ such that $U[f_2, B] = \lceil N/4 \rceil$. As before, the payoff of P2 is at most $\lceil N/2 \rceil$.

3 Optimal Strategies in the 2-Round Game

In this section, we show how the optimal payoffs of the players change (as their location varies over the interval $[A, B]$) in the 2-round game, and use this to prove Theorem 1.3.

To this end, for any two facilities $a, b \in L$ (belonging to P1 or P2), such that $U[a, b] > 0$, denote by $cov([a, b])$ the maximum number of users a new facility can serve by placing a single point in the interval (a, b). Note that if the sorted order of the points in U are given, then by scanning a segment of length $\frac{1}{2}|a - b|$ in the interval $[a, b]$, we can compute $cov([a, b])$ easily in $O(U[a, b]) = O(N)$ time [3].

3.1 Round 2

Given any placement f_1 by P1 and s_1 by P2 in round 1, the *landscape of the payoff of P1 in round 2* is the function $\eta : L = [A, B] \to \mathbb{R}$,

$$\eta(f_2) := \min_{s_2 \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\}), \tag{3.1}$$

which is the optimal payoff when P1 places at f_2 in round 2. Similarly, given a placement f_1 by P1 and s_1 by P2 in round 1, and f_2 by P1 in round 2, the *landscape of the payoff of P2 in round 2* is the function $\theta : L = [A, B] \to \mathbb{R}$, where $\theta(s_2) := \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\})$. It is easy to see that both the landscape functions are piecewise constant over L, and the number of pieces will be referred to as the *complexity* of the landscape function. Once we determine the complexity (that is, the number of pieces and the location of the pieces) of the payoff functions, the optimal payoff of the players in round 2 can be determined by computing the value of the function $\eta(\cdot)$ (or $\theta(\cdot)$) in every piece.

Proposition 3.1. *The optimal strategies of P2 and P1 in round 2 can be computed in $O(N)$ and $O(N^2)$ times, respectively.*

Proof. It is easy to see that $\theta(\cdot)$ is piecewise constant with $O(1)$-pieces, that is, landscape of P2 in round 2 is of constant complexity. Since $\theta(s_2)$ at any point $s_2 \in L$ can be computed in $O(N)$ time, this implies that the optimal payoff of P2 can be found in $O(N)$ time.

Now, we discuss the strategy of P1 in round 2. Hereafter, we assume $s_1 > f_1$, with the other case done similarly. Define $\mathcal{D} = \{2u_a - s_1 : u_a \in U\} \cap L$, and set $\mathcal{E} = \mathcal{D} \cup U$. Now, let $f_2 \notin \mathcal{E}$ be optimal placement by P1 in round 2, and consider the following three cases:

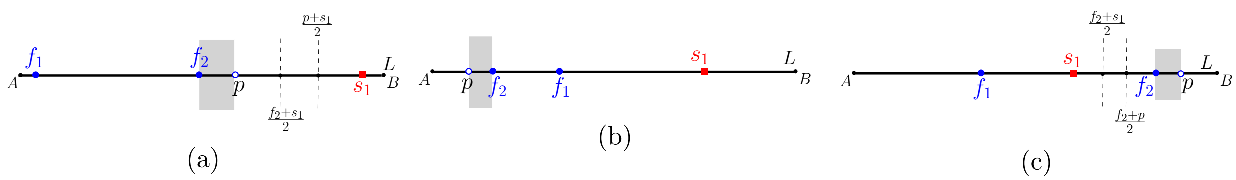

Figure 3: Optimal strategy of P1 in round 2: (a) $f_2 \in [f_1, s_1]$, (b) $f_2 \in [A, f_1]$, and (c) $f_2 \in [s_1, B]$. The regions on L shown in grey have no point of $\mathcal{E} = \mathcal{D} \cup U$ in their interiors.

Case **1:** $f_2 \in [f_1, s_1]$. Let $p \in \mathcal{E}$ be the closest point in $\mathcal{E}$ to the right of f_2. Observe that, given the placements f_1, s_1, f_2, if P2 plays optimally then the payoff of P1 is

$$\begin{aligned}
&\min_{x \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) \\
&= U[A, f_2] + U\left[f_2, \frac{f_2 + s_1}{2}\right] - \max\left\{U[A, f_1], cov(f_1, f_2), U\left[f_2, \frac{f_2 + s_1}{2}\right]\right\} \\
&\geq U[A, p] + U\left[p, \frac{p + s_1}{2}\right] - \max\left\{U[A, f_1], cov(f_1, p), U\left(p, \frac{p + s_1}{2}\right)\right\},
\end{aligned}$$

since $cov(f_1, f_2) < cov(f_1, p)$ and $[\frac{f_2+s_1}{2}, \frac{p+s_1}{2})$ does not contain any user (by definition of the set $\mathcal{D}$).

Case **2:** $f_2 \in [A, f_1]$. Let $p \in U$ be the closest user to the left of f_2. Then

$$\begin{aligned}\min_{x\in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) &= U\left[A, \frac{f_1+s_1}{2}\right] - \max\left\{U[A, f_2], cov(f_2, f_1), U\left[f_1, \frac{f_1+s_1}{2}\right]\right\} \\ &\geq U\left[A, \frac{f_1+s_1}{2}\right] - \max\left\{U[A, p), cov(p, f_1), U\left[f_1, \frac{f_1+s_1}{2}\right]\right\},\end{aligned}$$

using $U[A, p) = U[A, f_2]$ and $cov(f_1, f_2) \leq cov(f_1, p)$.

Case **3:** $f_2 \in [s_1, B]$. In this case, let $p \in U \cap [s_1, B]$ be the closest point in U to the right of f_2. Then using $U[f_2, B] = U(p, B]$ and $U\left(\frac{f_1+s_1}{2}, \frac{s_1+f_2}{2}\right) \leq U\left(\frac{f_1+s_1}{2}, \frac{s_1+p}{2}\right)$ gives,

$$\begin{aligned}&\min_{x\in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) \\ &= N - U\left(\frac{f_1+s_1}{2}, \frac{s_1+f_2}{2}\right) - \max\left\{U[A, f_1], U\left[f_1, \frac{f_1+s_1}{2}\right], U\left[\frac{s_1+f_2}{2}, f_2\right], U[f_2, B]\right\}.\end{aligned}$$

If

$$U\left[\frac{s_1+f_2}{2}, f_2\right] \neq \arg\max\left\{U[A, f_1], U\left[f_1, \frac{f_1+s_1}{2}\right], U\left[\frac{s_1+f_2}{2}, f_2\right], U[f_2, B]\right\},$$

then $U\left(\frac{s_1+p}{2}, p\right) \leq U(\frac{s_1+f_2}{2}, f_2)$, and moving f_2 to p does not change $\min_{x\in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\})$. Otherwise, the maximum above is attained at $U\left[\frac{s_1+f_2}{2}, f_2\right]$, in which case

$$\min_{x\in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) = N - U\left(\frac{f_1+s_1}{2}, f_2\right] = N - U\left(\frac{f_1+s_1}{2}, p\right),$$

that is, the payoff is again unchanged, when f_2 moves to p.

The cases above show that for computing the optimal strategy of P1 in round 2, one has to compute $\eta(f_2)$ at $O(N)$ points. As it takes $O(N)$ time to compute $\eta(f_2)$ at single point, the optimal optimal strategy of P1 in round 2 can be computed in $O(N^2)$ time. □

3.2 Round 1

The payoff landscapes become increasingly complicated as the number of rounds increases. Here, we compute the landscape of the payoffs in round 1 of the 2-round game: Given a placement f_1 by P1 in round 1, the *landscape of the payoff of P2 in round 1* is the function $\psi : L \to \mathbb{R}$:

$$\psi(s_1) := \min_{f_2\in L} \max_{s_2\in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\}), \tag{3.2}$$

which is the payoff of P2 when it places the first facility at s_1, and both P1 and P2 place their respective facilities optimally in the second round. Similarly, the *landscape of the payoff of P1 in round 1* is

$$\varphi(f_1) := \min_{s_1\in L} \max_{f_2\in L} \min_{s_2\in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\}). \tag{3.3}$$

As in round 2, the functions $\psi(\cdot)$ and $\varphi(\cdot)$ are piecewise linear, and once we determine the location of the pieces, the optimal payoffs of the players in round 1 can be determined by computing the values of the functions in every piece.

Proposition 3.2. *The optimal strategies of P2 and P1 in round 1 can be computed in $O(N^5)$ and $O(N^9)$ times, respectively.*

The proposition is above proved below. This, combined with Proposition 3.2 above, completes the proof of Theorem 1.3.

Proof of Proposition 3.2: We begin the optimal strategy of P2 in round 1. Given the placement of f_1 by P1, P2 can place its first facility either in $[A, f_1)$ or (f_1, B). Here, we analyze the case where P2 places its first facility in the interval (f_1, B). The other case can be done similarly. After P2 places at s_1 in round 1, in round 2 P1 can place its facility in either one of the three intervals $[A, f_1]$, $[f_1, s_1]$ or $[s_1, B]$. Therefore, the minimum in (3.2) can be written as:

$$\psi(s_1) := \min\left\{\psi_1(s_1), \psi_2(s_1), \psi_3(s_1)\right\},$$

where $\psi_1(s_1) := \min_{f_2 \in [A, f_1]} \max_{s_2 \in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\})$, $\psi_2(s_1) := \min_{f_2 \in [f_1, s_1]} \max_{s_2 \in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\})$, and $\psi_3(s_1) := \min_{f_2 \in [s_1, B]} \max_{s_2 \in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\})$.

Lemma 3.1. *The functions ψ_1, ψ_2, $\psi_3 : [f_1, B] \to \mathbb{Z}_+ \cup \{0\}$ are piecewise constant, with $O(N)$, $O(N^3)$, and $O(N^2)$ pieces, respectively.*

Proof. We begin with ψ_1. In this case (referring to Figure 3(b)),

$$\psi_1(s_1) = U[s_1, B] + U\left(\frac{f_1 + s_1}{2}, s_1\right] + \min_{f_2 \in [A, f_1]} \max\left\{U[A, f_2], cov([f_2, f_1]), U\left[f_1, \frac{f_1 + s_1}{2}\right)\right\}.$$

Therefore, the set of points where $\psi_1(\cdot)$ changes its values is $\mathcal{A}_1 \cup \mathcal{B}_1$, where $\mathcal{A}_1 = \{2u_a - f_1 : u_a \in u_a \in [f_1, B) \cap U\} \cap L$ and $\mathcal{B}_1 = [f_1, B) \cap U$. To see this note that if $s_1 \in \mathcal{A}_1$, then $\frac{f_1+s_1}{2} \in U$, and therefore $U[f_1, \frac{f_1+s_1}{2}]$ and, hence $\psi_1(s_1)$, changes in the neighborhood of this point. Similarly, $U[s_1, B]$, and, hence $\psi_1(s_1)$, changes, in the neighborhood of $s_1 \in \mathcal{B}_1$. As, $|\mathcal{A}_1 \cup \mathcal{B}_1| = O(N)$, the function ψ_1 can have at most $O(N)$ pieces.

Next, we look at ψ_2. In this case (referring to Figure 3(a)),

$$\psi_2(s_1) = \min_{f_2 \in [f_1, s_1]} \left\{U\left(\frac{f_2 + s_1}{2}, B\right] + \max\left\{U[A, f_1], cov(f_1, f_2), U\left[f_2, \frac{f_2 + s_1}{2}\right)\right\}\right\}.$$

Let $\hat{f}_2 \in [f_1, s_1]$ be a point which attains the minimum above. Then as s_1 varies over L, $\psi_2(s_1)$ changes when $\frac{\hat{f}_2+s_1}{2} \in U$. We now argue that it suffices to assume (a) $\hat{f}_2 \in U$, or (b) $\hat{f}_2 = f_1 + 2|u_a - u_b|$, for some $u_a, u_b \in U$. Then there are two cases:

- Let $f_2' < \hat{f}_2$ be such that $cov(f_1, f_2') = U[u_a, u_b]$ and $f_2' = f_1 + 2|u_a - u_b|$, where $u_a, u_b \in U$ is such that $cov(f_1, \hat{f}_2) = U[u_a, u_b]$. (Given $x, y \in L$, there exists $u_a, u_b \in U \cap [x, y]$ such that $cov(x, y) = U[u_a, u_b]$, because any interval $I \subset [x, y]$ which attains $cov(x, y) = |U \cap I|$ can be shrunk to an interval $[u_a, u_b]$ such that $cov(x, y) = U[u_a, u_b]$.) If $U[f_2', \hat{f}_2]$ is empty, then the value of $\psi_2(s_1)$ remain unchanged, when $\hat{f}_2$ is replaced by f_2', that is, (b) holds.
- Otherwise, move $\hat{f}_2$ to the closest user u' to the left of $\hat{f}_2$ in $[f_2', \hat{f}_2]$. Then $cov(f_1, \hat{f}_2) = cov(f_1, u')$, and, therefore, $\psi_2(s_1)$ remain unchanged, that is, (a) holds in this case.

Therefore, the set of points where $\psi_2(\cdot)$ changes is contained in $\mathcal{A}_2 \cup \mathcal{B}_2$, where $\mathcal{A}_2 := \bigcup_{u \in U}\{2u - f :$ such that $f = f_1 + 2|u_a - u_b|$ for some $u_a, u_b \in U\}$ and $\mathcal{B}_2 = \bigcup_{u \in U}\{2u - f : f \in U\}$. The result about $\psi_2(\cdot)$ follows by noting that $|\mathcal{A}_2 \cup \mathcal{B}_2| = O(N^3)$, as required.

Finally, we consider $\psi_3(s_1)$. Again, referring to Figure 3(c), it follows that

$$\psi_3(s_1)$$
$$= \min_{f_2 \in [s_1, B]} \left\{ U\left(\frac{f_1 + s_1}{2}, \frac{s_1 + f_2}{2}\right) + \max\left\{ U[A, f_1], U\left[f_1, \frac{f_1 + s_1}{2}\right), U\left(\frac{s_1 + f_2}{2}, f_2\right), U(f_2, B] \right\} \right\}.$$

This can change, for points in the neighborhood of s_1 such that either (a) $\frac{s_1 + f_1}{2} \in U$ or (b) $\frac{f_2 + s_1}{2} \in U$. The set of points where (a) happens is $\mathcal{A}_3 := \{2u - f_1 : u \in [f_1, B) \cap U\}$. For (b) note that the optimal placement of f_2 can be obtained by checking in the neighborhood of users in $[s_1, B]$ (recall the third case in the proof of Proposition 3.1). Therefore, the set of points where s_1 can be placed such that if $f_2 \in [s_1, B] \cap U$, then $\frac{f_2 + s_1}{2} \in U$, is contained in $\mathcal{B}_3 := \{2u_a - u_b : u_a, u_b \in [f_1, B) \cap U\}$. This implies that the function $\psi_3(s_1)$ can have at most $O(N^2)$ pieces, as $|\mathcal{A}_3 \cup \mathcal{B}_3| = O(N^2)$. □

Note that computing the optimal payoff of P2 at a point in round 1 requires $O(N^2)$ time (Proposition 3.1). Then by the above lemma the optimal strategy of P2 in round 1 can be computed in $O(N^5)$ time.

Now, we consider the strategy of P1 in round. Recall the definition of the landscape of the payoff of P1 in round 1 from (3.3). Then

$$\varphi(f_1) = \min_{s_1 \in L} \max_{f_2 \in L} \min_{s_2 \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\}) = \min\{\varphi_1(f_1), \varphi_2(f_1)\},$$

where $\varphi_1(f_1) := \min_{s_1 \in [A, f_1]} \max_{f_2 \in L} \min_{s_2 \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\})$, that is, P2 is restricted to place in the interval $[A, f_1]$ in round 1, and, $\varphi_2(f_1) := \min_{s_1 \in [f_1, B]} \max_{f_2 \in L} \min_{s_2 \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\})$, where P2 is restricted to place in the interval $[f_1, B]$ in round 1. Both $\varphi_1(f_1)$ and $\varphi_2(f_1)$ are piecewise constant functions as f_1 varies in L. In the following, we will describe the complexity of the graph of $\varphi_2(f_1)$, and can be $\varphi_1(f_1)$ done similarly.

Observation 3.1. *Let f_1, s_1 be placement of facilities by P1 and P2 in round 1, such that $U[f_1, s_1] > 0$. Now, if the optimal placement of P1 in round 2 is at $f_2 \in (A, f_1)$, then it is possible to move s_1 immediately to the right of an user in U, without decreasing the payoff of P2.*

Proof. Let $u'_1 < u'_2 < \ldots < u'_N$ be the sorted order of the users in U, such that $s_1 \in (u'_j, u'_{j+1})$. If we move s_1 immediately to the right of u'_j (which we denote by s'_1), then $U[s_1, B] = U[s'_1, B]$, but the point $\frac{f_1 + s_1}{2}$ moves left to $\frac{f_1 + s'_1}{2}$, which may lead to f_2 to move to a point f'_2 in (f_1, s'_1). Then the payoff of P2 in round 2, when f_2 moves to f'_2 (after s_1 moves to s'_1) is

$$U\left(\frac{f'_2 + s'_1}{2}, B\right] + \max\left\{U[A, f_1], cov(f_1, f'_2), U\left[f'_2, \frac{f'_2 + s'_1}{2}\right]\right\}$$
$$\geq U\left(\frac{f'_2 + s_1}{2}, B\right] + \max\left\{U[A, f_1], cov(f_1, f'_2), U\left[f'_2, \frac{f'_2 + s_1}{2}\right]\right\}$$
$$\geq U\left(\frac{f_1 + s_1}{2}, B\right] + \max\left\{U[A, f_2], cov(f_1, f_2), U\left[f_1, \frac{f_1 + s_1}{2}\right]\right\},$$

where the last inequality uses the assumption that f_2 is the optimal placement of P1 in round 2 (hence the payoff of P2 when P1 places at f_2' instead of at f_2 will be larger.) This shows that the optimal location of P1 in round 2 remains unchanged when s_1 moves to s_1', completing the proof of the lemma. □

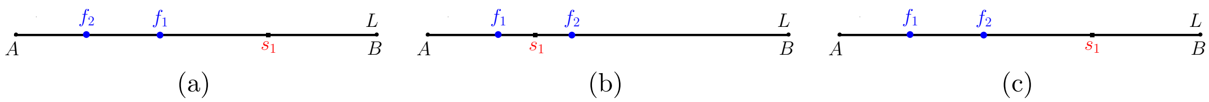

Figure 4: Optimal strategy of P1 in round 1: (a) *Case* 1, (b) *Case* 2, (c) *Case* 3.

Now, we compute the complexity of $\varphi_2(\cdot)$. Recall, we are assuming $s_1 \in (f_1, B]$, and depending on the location of f_2 there are 3 cases:

Case **1:** $f_2 < f_1 < s_1$. In this case, the payoff of P1 is

$$N - U\left[\frac{f_1+s_1}{2}, B\right] - \max\left\{U[A, f_2), cov([f_2, f_1]), U\left[\frac{f_1+s_1}{2}, B\right]\right\}.$$

In this case, as f_1 moves along L, the payoff above changes when either $cov([f_2, f_1])$ changes, or $\frac{f_1+s_1}{2}$ or f_2 passes through an user. Note that, in this case, we can essentially assume $s_1 \in U$ (by Observation 3.1 it suffices to check immediately to left or right of users). Therefore, the set of points f_1 for which $U[f_1, \frac{f_1+s_1}{2}]$ changes can be expressed as $\mathcal{A}_1 = \{2u_a - u_b : u_a, u_b \in U\}$. Next, we try to find the set of f_1 for which $cov([f_2, f_1])$ changes. Note that for each possible value of $cov([f_2, f_1])$, there is an interval $[u_a, u_b]$, where $u_a, u_b \in U$, such that $cov([f_2, f_1]) = U[u_a, u_b]$. Moreover, it suffices to assume that in round 2, P1 places $f_2 \in U$ (recall the second case in the proof of Proposition 3.1). Therefore, considering each possible of placement of f_2 on an user and all pair of users in U, the set of points f_1 for which $cov([f_2, f_1])$ changes is contained in $\mathcal{B}_1 = \{u_a \pm 2|u_b - u_c| : u_a, u_b, u_c \in U\}$. As $|\mathcal{A}_1 \cup \mathcal{B}_1| = O(N^3)$, the proof of this case is complete.

Case **2:** $f_1 < s_1 < f_2$. In this case, the payoff of P1 is

$$N - U\left(\frac{f_1+s_1}{2}, \frac{s_1+f_2}{2}\right) - \max\left\{U[A, f_1], U\left[f_1, \frac{f_1+s_1}{2}\right), U\left(\frac{s_1+f_2}{2}, f_2\right), U(f_2, B]\right\}.$$

To begin with, note that it suffices to assume that $f_2 \in U$ (by the third case in the proof of Proposition 3.1). Now, observe that for each possible value of $U(\frac{f_1+s_1}{2}, \frac{s_1+f_2}{2})$ there exists $u_a, u_b \in U$ such that $U(\frac{f_1+s_1}{2}, \frac{s_1+f_2}{2}) = U[u_a, u_b]$. Therefore, given a placement $f_2 \in U$, we can move s_1 either to an user or to a point such that $\frac{f_1+s_1}{2} \in U$ keeping the payoff of P1 unchanged, and given the location of s_1 we can move f_1 either to an user or to a point such that $\frac{s_1+f_2}{2} \in U$ (again keeping the payoff of P1 unchanged). This means the set of possible values of s_1 is contained in $\mathcal{B}_2 := \{2u_a - u_b : u_a, u_b \in U\}$ (since $f_2 \in U$), and the set of possible values of f_1 is contained in $\mathcal{A}_2 := \{2u_c - u' : \text{ where } u_c \in U \text{ and } u' \in \mathcal{B}_2\}$, which satisfies $|\mathcal{A}_2| = O(N^3)$. (Note that both the sets $\mathcal{A}_2$ and $\mathcal{B}_2$ contain the user set U.)

Case **3:** $f_1 < f_2 < s_1$. In this case, the payoff of P1 is

$$N - U\left[\frac{f_2 + s_1}{2}, B\right] - \max\left\{U[A, f_1], cov([f_2, f_1]), U\left[f_2, \frac{f_2 + s_1}{2}\right]\right\}.$$

As before, for each possible value of $cov([f_2, f_1])$, there is an interval $[u_a, u_b]$, where $u_a, u_b \in U$, such that $cov([f_2, f_1]) = U[u_a, u_b]$. As f_1 moves along L, the payoff above will change if one the following 3 cases happen:

- f_1 coincides with an user: In this case, the set of possible choices of f_1 is just U.
- f_2 coincides with an user: In this case, the set of possible choices of f_1 is contained in $\mathcal{A}_{31} := \{u_c - 2|u_a - u_b|, \text{ for } u_a, u_b, u_c \in U\}$, using $f_2 - f_1 = 2|u_a - u_b|$ and $f_2 \in U$.
- $\frac{f_2+s_1}{2}$ coincides with a user: In this case, we can move s_1 to the closest user to its left and f_2 to its right so the midpoint $\frac{f_2+s_1}{2}$ remain unchanged (otherwise, one of the previous two cases happen), without changing the payoff of P1. As s_1 and $\frac{f_2+s_1}{2}$ now both coincide with users, this gives $\mathcal{B}'_3 = \{2u_c - u_d : u_c, u_d \in U\}$ choices for f_2, and $\mathcal{A}_{32} = \{f - 2|u_a - u_b| : f \in \mathcal{B}'_3 \text{ and } u_a, u_b \in U\}$ choices for f_1.

Therefore, the set of possible choices of f_1 is contained in $U \cup \mathcal{A}_{31} \cup \mathcal{A}_{32}$, with $|U \cup \mathcal{A}_{31} \cup \mathcal{A}_{32}| = O(N^4)$, completing the proof. □

The cases above show that it is enough to compute $\varphi(f_1)$ in $O(N^4)$ points, which implies the optimal strategy of P1 in round 1 can be found in $O(N^9)$ time (since the optimal strategy of P2 in round 1 can be computed in $O(N^5)$ time). This completes the proof of Proposition 3.2 □

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