The 1-Dimensional Discrete Voronoi Game

Aritra Banik† Bhaswar B. Bhattacharya† Sandip Das§ Sreeja Das¶

Abstract

In this paper we study the discrete version of the 1-dimensional (continuous) Voronoi game introduced by Ahn et al. [1]. The discrete Voronoi game in dimension 1, consists of two competing players P1 and P2 and a set of N users placed on a line-segment. The players alternately place one facility each on the line-segment for R-rounds, where the objective is to maximize their own total payoffs. We prove bounds on the worst-case (over the arrangement of the N users) payoffs of the two players, and show that they are often tight. We also compute the complexities of the optimal payoff functions and discuss algorithms for finding the optimal strategies of the players, in 2-round game.

1 Introduction

Competitive facility location is concerned with the favorable placement of facilities by competing market players [12, 13]. It goes back to the 1929 seminal paper by Hotelling [15] which introduced the competitive facility location problem when the users were placed uniformly on a line segment. Facilities and users are generally modeled as points in a pre-specified arena (generally a subset of $\mathbb{R}^1/\mathbb{R}^2$). The set of users (demands) is a subset of the arena, which can be either discrete, consisting of finitely many points, or continuous, that is, a region where every point is considered to be a user. We assume that the facilities are equally equipped in all respects, and a user always avails the service from its nearest facility. Consequently, each facility has its service zone, consisting of the set of users that are served by it, and the goal is to find placement of facilities which maximize the cardinality or the area of its service zone, depending on whether the demand region is discrete or continuous, respectively. For a recent survey on the applications of competitive facility location in economics and operations research, refer to [11].

Ahn et al. [1] introduced a game-theoretic analogue of such problems for 1-dimensional arenas (line-segment/circle) with continuous demands. The game consists of the 2 players P1 and P2 alternately placing disjoint set of facilities in the arena. In this case, the payoff of player P1/P2 is the area of the region that is closer to the facilities owned by P1/P2 than to the other player, and
the player which finally owns the larger area is the winner of the game. They showed that when the
players place one facility each for $R$-rounds, the second player always has a winning strategy that
guarantees a payoff of $1/2 + \varepsilon$, with $\varepsilon > 0$. However, the first player can force $\varepsilon$ to be arbitrarily
small.

In this paper, we study the analogous version of the game for discrete demand regions: Given
a positive integer $R \geq 1$, the 1-dimensional $R$-round discrete Voronoi game consists of two players
P1 and P2 and a set $U$ of $N$ users on a line-segment $L \subset \mathbb{R}$. The players alternately place one
facility each for $R$-rounds, with the objective to maximize their own total payoffs, where the payoff
of P1/P2 is the cardinality of the set of points in $U$ which are closer to a facility owned by P1/P2
than to every facility owned by P2/P1. To define this more formally, we introduce some notation:
Given a set $F \subset \mathbb{R}$ of facilities, define for every $f \in F$,
$$U(f, F) = \{u_a \in U : |u_a - f| < |u_a - h|, \text{ for all } h \in F \setminus \{f\}\},$$
the set of users which are closest to $f$. Then, for any placement of facilities $S_1$ by P1 and $S_2$ by P2, the payoff of P2 is $P_2(S_1, S_2|U) = |\bigcup_{f \in S_2} U(f, S_1 \cup S_2)|$. Similarly, the payoff of P1,
$P_1(S_1, S_2|U) = |U| - P_2(S_1, S_2|U)$. Note that this definition implies that if an user is equidistant
from a facility in $S_1$ and another facility in $S_2$, then it contributes to the payoff of P1, that is, ties
are broken in favor of P1. Given a set of users $U$, denote by $\eta_2(R|U)$ the maximum possible payoff
P2 can attain against any adversarial strategy of P1, when the game is played for $R$-rounds.

Theorem 1.1. For any $R \geq 2$, the following hold:

(a) For any placement $U$ of $N$ users in the segment $L$, $\eta_2(R|U) \geq N \left(\frac{\sqrt{8(R+1)} - 1}{2R}\right) - (R - 1)$.

(b) For any placement $U$ of $N$ users in the segment $L$, $\eta_1(R|U) \geq \lfloor \frac{N}{R+1} \rfloor$. Moreover, there is a
placement $U_0$ of $N$ users such that $\eta_1(R|U_0) \leq \lceil \frac{N}{R+1} \rceil + R$.

The proof of the above theorem is given in Section 2. The lower bound on $\eta_1(R|U)$ follows by
a trivial pigeon-hole argument. However, quite surprisingly, it turns out there is an arrangement
of users for which this bound is attained (up to constant factors). In fact, the theorem gives the
exact asymptotics of the normalized worst-case payoff of P1, that is,
$$\lim_{N \to \infty} \inf_{U : |U| = N} \frac{\eta_1(R|U)}{N} = \frac{1}{R + 1},$$
where the infimum is taken over all arrangement of $N$ users on $L$. This shows how the arrangement
of users affect the payoff of P1, and is in sharp contrast with the continuous Voronoi game of Ahn
et al. [1], where P1 can always have a payoff which is arbitrarily close to 1/2.

Obtaining lower bounds on the payoff of P2 is more challenging. The bound in the theorem
above, shows that the normalized worst-case payoff of P2 is at least $1/2 \left(\sqrt{8(R+1)} - 1\right) = \Theta(1/\sqrt{R})$.
In the special case $R = 2$, this can be improved as follows:

1 We will assume that facilities are not allowed to overlap with themselves or with the set of users at any stage of
the game.
Corollary 1.2. For any placement \( U \) of \( N \) users in the segment \( L \), \( \eta_2(2|U) \geq \lceil N/2 \rceil \). Moreover, there is a placement \( U_0 \) of \( N \) users such that \( \eta_2(R|U_0) \leq \lfloor N/2 \rfloor \).

The above corollary shows that, irrespective of the arrangement of the users \( U \) on \( L \), \( P2 \) can only lose the 2-round game by at most a single user. This also implies that \( \lim_{N \to \infty} \frac{1}{N} \inf_{|U|=N} \eta_2(R|U) = \frac{1}{2} \), that is, \( P2 \) can always asymptotically tie the game, irrespective of the placement of the users on \( L \). Whether \( P2 \) can always asymptotically tie the game, for any \( R \geq 3 \), remains open.

Finally, in Section 3 we compute the landscape of the optimal payoff function in the 2-round game. This involves finding how the payoffs of the players change, assuming each player plays optimally, that is, to maximize their own eventually payoffs, in the subsequent rounds.

1.1 Related Work

Dehne et al. [10] studied a competitive facility location problem for continuous demand regions in \( \mathbb{R}^2 \), where the problem is to find a new point \( q \) amidst a set of \( n \) existing points \( F \) such that the Voronoi region of \( q \) is maximized. They showed that when the points in \( F \) are in convex position, the area function has only a single local maximum inside the region where the set of Voronoi neighbors do not change. For the same problem, Cheong et al. [8] gave a near-linear time algorithm that determines the location of the new optimal point approximately, when the points in \( F \) are in general position. In the discrete user case, the analogous problem is to place a set of new facilities amidst a set of existing ones such that the number of users served by the new facilities is maximized [6, 7].

Cheong et al. [9] studied the 1-round (continuous) Voronoi game in \( \mathbb{R}^2 \) for a square-shaped demand region, which was later extended by Fekete and Meijer [14] to rectangular demand regions. Here, \( P1 \) followed by \( P2 \), places \( m \) facilities in the demand region, and the player with the larger Voronoi area wins the game. Recently, variants of these games when the demand region is a graph equipped with the shortest-path distance [2] has been studied.

Banik et al. [3] studied the one-round discrete Voronoi game in \( \mathbb{R} \), where, given a set \( U \) of \( N \) users on a line, \( P1 \) chooses a set of \( m \) facilities, following which \( P2 \) chooses another disjoint set of \( m \) facilities, and the objective of both the players is to maximize the number of users they serve. The authors showed that if the sorted order of the points in \( U \) along the line is known, then the optimal strategy of \( P2 \), given any placement of facilities by \( P1 \), can be computed in \( O(N) \) time, and the optimal strategy of \( P1 \) can be computed in \( O(N^{m-\lambda_m}) \) time, where \( 0 < \lambda_m < 1 \), is a constant depending only on \( m \). In 2-dimensions, exact and approximation algorithms for the discrete Voronoi game in some special cases, were recently obtained in [4] and [5], respectively.

2 Proofs

Throughout this section we will assume that the playing arena is the interval \( L = [A,B] \). Moreover, for any two points \( a, b \in L \), with \( a < b \), we will denote by \( U[a,b] = |[a,b] \cap U| \), the number of users in \( U \) in the interval \( [a,b] \). The definition is naturally modified when one or both of the endpoints of the interval are open.

\(^3\)In the 1-round game, \( P2 \) can trivially tie the game, with a payoff of at least \( \lceil N/2 \rfloor \) by placing its facility immediately next to the facility of \( P1 \), either to the right or to the left, depending on which side has more users in \( U \).
2.1 Proof of Theorem 1.1

We begin the following simple observation:

**Observation 2.1.** Given any placement \( \mathcal{F} \) of \( M \) facilities in an interval \( L_0 = [A_0, B_0] \subseteq \mathbb{L} \), there exists \( s \in L_0 \) such that at least \( \frac{1}{2M} |U \cap L_0| \) users in \( U \cap L_0 \) are closer to \( s \) than to the \( M \) users in \( \mathcal{F} \).

**Proof.** Let \( \mathcal{F} = \{ f_1, f_2, \ldots, f_M \} \) such that \( f_1 < f_2 < \cdots < f_M \). If either \( U[A_0, f_1] \geq \frac{1}{2M} |U \cap L_0| \) or \( U[f_M, B_0] \geq \frac{1}{2M} |U \cap L_0| \), then by placing \( s \) immediately to the left of \( f_1 \) or right of \( f_M \), respectively, the result follows.

Therefore, assume that \( U[A_0, f_1] < \frac{1}{2M} |U \cap L_0| \) and \( U[f_M, B_0] < \frac{1}{2M} |U \cap L_0| \). Then, by the pigeonhole principle, there exists \( 1 \leq j \leq M - 1 \) such that \( U[f_j, f_{j+1}] \geq \frac{1}{M} |U \cap L_0| \). Therefore, placing \( s \) immediately to the right\(^3\) of \( f_j \) or left \( f_{j+1} \) is possible for \( s \) to get to at least \( \frac{1}{2M} |U \cap L_0| \) users in \( U \cap L_0 \).

![Figure 1: Proof of Theorem 1.1: Strategy of P2 in the \( R = 5 \) round game, when \( b = 2 \) (where \( b \) is the number facilities of P1 to the left of \( s_4 \)).](image)

Now, we propose a strategy for P2 which gives the required lower bound in Theorem 1.1. In the first \( R - 1 \) rounds, P2 places its facilities at \( s_1, s_2, \ldots, s_{R-1} \), respectively, where \( s_1, s_2, \ldots, s_{R-1} \) are chosen such that \( U[A, s_1] = |N/2R| \) and \( U[s_j, s_{j+1}] = |N/2R| \), for \( 1 \leq j \leq R - 2 \). Suppose, after all the \( R \) rounds there are \( 1 \leq b \leq R \) facilities of P1 in the interval \( U[A, s_{R-1}] \) and \( R - b \) in the interval \( U[s_{R-1}, B] \). If \( b = R \), then all points in the interval \( U[s_{R-1}, B] \) belong to P2, and \( \eta_2(R|U) \geq N - \lfloor \frac{N}{2R} \rfloor (R - 1) \geq \frac{R+1}{2R} N \). Otherwise, \( 1 \leq b \leq R - 1 \) and by Lemma 2.1 P2 can chose \( s_R \in [s_{R-1}, B] \) such that P2 gets at least \( \frac{1}{2(R-b)} |[s_{R-1}, B]| = \frac{N-\lfloor \frac{N}{2R} \rfloor (R-1)}{2(R-b)} \) users. Moreover, since at least \( R - b - 1 \) of the intervals \( [A, s_1], [s_1, s_2], \ldots, [s_{R-2}, s_{R-1}] \) contains no point from P1, the total payoff P2 is at least

\[
\eta_2(R|U) \geq (R - b - 1) \left\lfloor \frac{N}{2R} \right\rfloor + \frac{N - \lfloor \frac{N}{2R} \rfloor (R - 1)}{2(R-b)}
\]

\(^3\)Throughout the paper, we will use the phrase ‘a point \( b \) is placed immediately to the right/left of another point \( a \)’ to mean that \( b = a \pm \varepsilon \), where \( \varepsilon > 0 \) is chosen to be arbitrarily small. In fact, choosing any \( \varepsilon < \frac{1}{1000} \min_{0 \leq j \leq N} |u_{j+1} - u_j| \), where \( u_0 = A \) and \( u_N = B \), would suffice for our purpose.
\[
\begin{align*}
\geq & (R - b - 1) \left( \frac{N}{2R} \right) + N - \frac{N}{2R} (R - 1) - (R - b - 1) \quad \text{(using } x - 1 \leq [x] \leq x) \\
\geq & \frac{N}{2R} \left( R - b + \frac{2(R+1)}{R - b} \right) - \frac{N}{2R} - (R - 1) \\
\geq & N \left( \frac{\sqrt{8(R+1)} - 1}{2R} \right) - (R - 1),
\end{align*}
\]

where the last step uses the fact that the function \( f(x) := x + \frac{2(R+1)}{x} \) is minimized at \( x = \sqrt{2(R+1)} \). This completes the proof of part (a).

Next, we prove part (b). The strategy for P1 which gives the required lower bound on the payoff of P1 is as follows: In the \( R \)-rounds P1 places its facilities at \( f_1, f_2, \ldots, f_R \), respectively, such that \( U[A, f_1] \geq \frac{N}{R+1} \), \( U[f_R, B] \geq \frac{N}{R+1} \) and \( U[f_j, f_{j+1}] \geq \frac{N}{R+1} \), for \( 1 \leq j \leq R - 1 \). By the pigeonhole principle, for any placement of \( R \) facilities by P2, at least one of these \( R + 1 \) intervals \([A, f_1], [f_1, f_2], \ldots, [f_{R-1}, f_R], [f_R, B] \) contains no points of P2, which implies \( \eta_1(R[U]) \geq \frac{N}{R+1} \).

To show this bound is attained, let \( U_0 = \{2, 4, \ldots, 2N\} \subset L_0 := [1, 2N+1] \). (Note that the points can be scaled to lie in any pre-specified line segment \( L \), if required.)

**Observation 2.2.** Let \( U_0 \) be as above and \( 1 \leq K, K' \leq N \). Suppose \( f \in (2^K, 2^{K+1}) \) and \( f' \in (2^{K+K'}, 2^{K+K'+1}) \) be two facilities of P1 such that the interval \((f, f')\) contains no other facilities of P1 and P2. Then P2 can place \( s \) immediately to the right of \( f \) and serve \( K' - 1 \) users in \( U[f, f'] \).

**Proof.** Note that \( \frac{1}{2}(s + f') > 2^{K-1} + \frac{f'}{2} \geq 2^{K-1} + 2^{K+K'-1} > 2^{K+K'-1} \). This implies, \( s \) serves \( K' - 1 \) users in the interval \((f, f')\).

This observation can be used to construct a strategy for P2. Depending on the placement \( f_1 \) of P1 in round 1, there are two cases:

- \( U_0[1, f_1] \leq \frac{N}{R+1} \): Then in round 1 P2 places \( s_1 \) immediately to the right of \( f_1 \). More generally, in the \( r \)-th round, where \( 1 \leq r \leq R \), given the placement \( f_1, f_2, \ldots, f_r \) by P1, P2 places \( s_r \) immediately to the right of \( f_r \). Let \( f'_1 < f'_2 < \ldots < f'_R \) be the sorted order of the facilities in P1 after \( R \) rounds are completed. This decomposes the segment \( L_0 \) into \( R + 1 \) intervals \([f'_0, f'_1], [f'_1, f'_2], [f'_2, f'_3], \ldots, [f'_{R-1}, f'_R], [f'_R, f'_{R+1}] \), where \( f'_0 = 1 \) and \( f'_{R+1} = 2N \). Now, by Observation 2.2, for every \( 1 \leq r \leq R, s'_r \) serves \( U_0[f'_r, f'_{r+1}] - 1 \), which implies that the payoff of P2 is at least

\[
\sum_{r=1}^{R} (U_0[f'_r, f'_{r+1}] - 1) = \sum_{r=0}^{R} U_0[f'_r, f'_{r+1}] - U_0[f'_0, f'_1] - R \geq N - \left\lfloor \frac{N}{R+1} \right\rfloor - R.
\]

Therefore, the payoff of P1 is at most \( \left\lfloor \frac{N}{R+1} \right\rfloor + R \).

- \( U[1, f_1] > \frac{N}{R+1} \): In this case, in round 1 P2 places \( s_1 \) such that \( U[1, s_1] = \left\lfloor \frac{N}{R+1} \right\rfloor \). Define \( S_0 := \{2 \leq j \leq R : f_j \in [1, s_1]\} \), that is, the rounds in which P1 places its facilities to the left of P1. Again there are two cases:

  - \( |S_0| \neq 0 \): Let \( j_1 < j_2 < \ldots < j_{|S_0|} \) be the elements in \( S_0 \) in increasing order. Then starting round 2 the strategy of P2 is as follows: If \( j \notin S_0 \), then place \( s_j \) immediately to
right of $f_j$. On the other hand, for $j_b \in S_0$, place $s_{j_b}$ immediately to right of $f_1$, but left $s_{b-1}$, for $1 \leq b \leq |S_0|$ (setting $s_0 := 2^N$). Then by Observation 2.2, it follows that the payoff of P1 is at most $\left\lceil \frac{N}{R+1} \right\rceil + R - |S_0| \leq \left\lceil \frac{N}{R+1} \right\rceil + R$ (since, at best P1 can get all the users in $[1, s_1]$ and one user each for the facilities in $\{f_j : j \notin S_0\}$).

- $|S_0| = 0$: Let $1 \leq r \leq R$, and $f_1, s_1, f_2, s_2, \ldots, f_r, s_r$ be the placements of the players P1 and P2 in the first $r$ rounds (recall that $s_1$ is chosen such that $U[1, s_1]$). Assume that $f_1^{(r)} < f_2^{(r)} < \ldots < f_r^{(r)} < f_{r+1}^{(r)} := B$ be the sorted order of the points on $L$. Now, suppose in the $(r+1)$-th round, P1 places at $f_{r+1}$ such that $f_{r+1} \in [f_j^{(r)}, f_{j+1}^{(r)}]$, for some $1 \leq j \leq r$. Again there are two cases:
  - $U[f_j^{(r)}, f_{j+1}^{(r)}] = \left\lceil \frac{N}{R+1} \right\rceil$: Then P2 places $s_r$ right next to $f_r$.
  - $U[f_j^{(r)}, f_{j+1}^{(r)}] < \left\lceil \frac{N}{R+1} \right\rceil$. Now, if $r$ is the first time this happens then P2 places $s_r$ immediately to the right of $f_1$. For every subsequently round when this happens, P2 places $s_r$ immediately to the right of $f_r$.

To see that this strategy works, let $f_1^{(R)} < f_2^{(R)} < \ldots < f_r^{(R)} < f_{r+1}^{(R)} := B$ be the sorted order of the points after the $R$ rounds. Note that $\sum_{i=1}^{R} U[f_i^{(R)}, f_{i+1}^{(R)}] \leq N - \left\lceil \frac{N}{R+1} \right\rceil$, which implies that there exists at least one $1 \leq j \leq R$ such that

$$U[f_j^{(R)}, f_{j+1}^{(R)}] < \left\lceil N - \left\lceil \frac{N}{R+1} \right\rceil \right\rceil \leq \left\lceil \frac{N}{R+1} \right\rceil.$$

Therefore, the strategy of P2 described above implies that there exists some $s_i^{(R)}$ immediately to right of $f_i^{(R)}$, for all $i \in \{1, 2, \ldots, R\}\{j_0\}$, where $j_0$ is such that $U[f_{j_0}^{(R)}, f_{j_0}^{(R)}] < \left\lceil \frac{N}{R+1} \right\rceil$. This implies that the total payoff P1 at most $\left\lceil \frac{N}{R+1} \right\rceil + R$.

### 2.2 Proof of Corollary 1.2

We begin by showing that there exists a strategy of P2 with payoff at least $\lceil N/2 \rceil$. To begin with note that for any placement $f_1$ by P1 either $U[A, f_1] \leq \lceil N/2 \rceil$ or $U[f_1, B] \leq \lceil N/2 \rceil$. Without loss of generality, we assume that $U[A, f_1] := K \leq \lceil N/2 \rceil$. Then, in round 1 P2 places $s_1 \in L$ such that $U[f_1, s_1] = \lceil N/2 \rceil$ (see Figure 2(a)). Now, consider the following 3 cases:

- $f_2 \in [A, f_1)$: In this case, placing $s_2$ immediately to the right of $f_1$, gives P2 a payoff of $N - K \geq \lceil N/2 \rceil$ (see Figure 2(a)).

- $f_2 \in (f_1, s_1)$: Then placing $s_2$ immediately to the left of $f_1$ gives P2 a payoff of $\lceil N/2 \rceil$ (see Figure 2(b)).

- $f_2 \in (s_1, B]$: In this case, again placing $s_2$ immediately to the right of $f_1$, gives P2 a payoff of $\lceil N/2 \rceil$ (see Figure 2(c)).

The argument above shows that $\eta_2(2U) \geq \lceil N/2 \rceil$. To show that this is attained, let $U_0$ be $N$ equally points in $L$, that is, $U_0 = \{A + \frac{j B - A}{N+1} : 1 \leq j \leq N\}$. In the first round P1 places at $f_1$, such that $U[A, f_1] = \lceil N/4 \rceil$. Now, depending on the placement $s_1$ of P1 in round 1, there are two cases:
Given any placement $f_1$ by P1 and $s_1$ by P2 in round 1, the landscape of the payoff of P1 in round 2 is the function $\eta : L = [A, B] \to \mathbb{R}$, 

$$\eta(f_2) := \min_{s_2 \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\}),$$

which is the optimal payoff when P1 places at $f_2$ in round 2. Similarly, given a placement $f_1$ by P1 and $s_1$ by P2 in round 1, and $f_2$ by P1 in round 2, the landscape of the payoff of P2 in round 2 is just $\theta(s_2) := \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\})$. It is easy to see that both the landscape functions are piecewise constant over $L$, and the number of pieces will be referred to as the complexity of the landscape function.

**Proposition 3.1.** The landscapes of the payoffs of P2 and P1 in round 2 have complexities $O(1)$ and $O(N)$, respectively.

**Proof:** It is easy to see that $\theta(\cdot)$ is piecewise constant with $O(1)$-pieces, that is, landscape of P2 in round 2 is of constant complexity.

Now, we discuss the strategy of P1 in round 2. Hereafter, we assume $s_1 > f_1$, with the other case done similarly. Define $\mathcal{D} = \{2u_a - s_1 : u_a \in U\} \cap L$, and set $\mathcal{E} = \mathcal{D} \cup U$. Now, let $f_2 \notin \mathcal{E}$ be optimal placement by P1 in round 2, and consider the following three cases:

- $s_1 < f_1$: In round 2, P2 places at $f_2 \in (s_1, B]$ such that $U[f_2, B] = \lfloor N/2 \rfloor$. Then, it is easy to see that for any placement $s_2$ of P2 in round 2, the payoff of P2 is at most $\lfloor N/4 \rfloor + \lceil \frac{N-2\lfloor N/4 \rfloor}{2} \rceil = \lfloor N/2 \rfloor$.

- $s_1 > f_1$ and $U[f_1, s_1] \leq \lfloor N/2 \rfloor$: As in the previous case, P2 places at $f_2 \in (s_1, B]$ such that $U[f_2, B] = \lfloor N/4 \rfloor$. Again, this guarantees that the payoff of P2 is at most $\lfloor N/2 \rfloor$.

- $U[f_1, s_1] > \lfloor N/2 \rfloor$: In round 2, P2 places at $f_2 \in (f_1, s_1)$ such that $U[f_2, B] = \lfloor N/4 \rfloor$. As before, the payoff of P2 is at most $\lfloor N/2 \rfloor$.

### 3 Landscape of the Payoffs in the 2-Round Game

In this section, we show how the optimal payoffs of the players change (as their location varies over the interval $[A, B]$) in the 2-round game. This can be used to devise algorithms for the optimal strategies of the players. To this end, for any two facilities $a, b \in L$ (belonging to P1 or P2), such that $U[a, b] > 0$, denote by $cov([a, b])$ the maximum number of users a new facility can serve by placing a single point in the interval $(a, b)$. Note that if the sorted order of the points in $U$ are given, then by scanning a segment of length $\frac{1}{2}|a-b|$ in the interval $[a, b]$, we can compute $cov([a, b])$ easily in $O(U[a, b]) = O(N)$ time [3].

#### 3.1 Round 2

Given any placement $f_1$ by P1 and $s_1$ by P2 in round 1, the landscape of the payoff of P1 in round 2 is the function $\eta : L = [A, B] \to \mathbb{R}$,

$$\eta(f_2) := \min_{s_2 \ L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\}),$$

which is the optimal payoff when P1 places at $f_2$ in round 2. Similarly, given a placement $f_1$ by P1 and $s_1$ by P2 in round 1, and $f_2$ by P1 in round 2, the landscape of the payoff of P2 in round 2 is just $\theta(s_2) := \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\})$. It is easy to see that both the landscape functions are piecewise constant over $L$, and the number of pieces will be referred to as the complexity of the landscape function.

**Proposition 3.1.** The landscapes of the payoffs of P2 and P1 in round 2 have complexities $O(1)$ and $O(N)$, respectively.

**Proof:** It is easy to see that $\theta(\cdot)$ is piecewise constant with $O(1)$-pieces, that is, landscape of P2 in round 2 is of constant complexity.

Now, we discuss the strategy of P1 in round 2. Hereafter, we assume $s_1 > f_1$, with the other case done similarly. Define $\mathcal{D} = \{2u_a - s_1 : u_a \in U\} \cap L$, and set $\mathcal{E} = \mathcal{D} \cup U$. Now, let $f_2 \notin \mathcal{E}$ be optimal placement by P1 in round 2, and consider the following three cases:
Figure 3: Optimal strategy of $P1$ in round 2: (a) $f_2 \in [f_1, s_1]$, (b) $f_2 \in [A, f_1]$, and (c) $f_2 \in [s_1, B]$. The regions on $L$ shown in grey have no point of $E = D \cup U$ in their interiors.

**Case 1:** $f_2 \in [f_1, s_1]$. Let $p \in E$ be the closest point in $E$ to the right of $f_2$. Observe that, given the placements $f_1, s_1, f_2$, if $P2$ plays optimally then the payoff of $P1$ is

$$
\min_{x \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) = U[A, f_2] + U \left[ f_2, \frac{f_2 + s_1}{2} \right] - \max \left\{ U[A, f_1], \cov(f_1, f_2), U \left[ f_2, \frac{f_2 + s_1}{2} \right] \right\}
$$

$$
\geq U[A, p] + U \left[ p, \frac{p + s_1}{2} \right] - \max \left\{ U[A, f_1], \cov(f_1, p), U \left( p, \frac{p + s_1}{2} \right) \right\},
$$

since $\cov(f_1, f_2) < \cov(f_1, p)$ and $[\frac{f_2 + s_1}{2}, \frac{p + s_1}{2}]$ does not contain any user (by definition of the set $D$).

**Case 2:** $f_2 \in [A, f_1]$. Let $p \in U$ be the closest user to the left of $f_2$. Then

$$
\min_{x \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) = U[A, f_2] + U \left[ f_1, \frac{f_1 + s_1}{2} \right] - \max \left\{ U[A, f_2], \cov(f_2, f_1), U \left[ f_1, \frac{f_1 + s_1}{2} \right] \right\}
$$

$$
\geq U \left[ A, \frac{f_1 + s_1}{2} \right] - \max \left\{ U[A, p], \cov(p, f_1), U \left[ f_1, \frac{f_1 + s_1}{2} \right] \right\},
$$

using $U[A, p] = U[A, f_2]$ and $\cov(f_1, f_2) \leq \cov(f_1, p)$.

**Case 3:** $f_2 \in [s_1, B]$. In this case, let $p \in U \cap [s_1, B]$ be the closest point in $U$ to the right of $f_2$. Then using $U[f_2, B] = U(p, B)$ and $U \left( \frac{f_1 + s_1}{2}, \frac{s_1 + f_2}{2} \right) \leq U \left( \frac{f_1 + s_1}{2}, \frac{s_1 + p}{2} \right)$ gives,

$$
\min_{x \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) = N - U \left[ \frac{f_1 + s_1}{2}, \frac{s_1 + f_2}{2} \right] - \max \left\{ U[A, f_1], U \left[ f_1, \frac{f_1 + s_1}{2} \right], U \left[ \frac{s_1 + f_2}{2}, f_2 \right], U[f_2, B] \right\}.
$$

If

$$
U \left[ \frac{s_1 + f_2}{2}, f_2 \right] \neq \arg \max \left\{ U[A, f_1], U \left[ f_1, \frac{f_1 + s_1}{2} \right], U \left[ \frac{s_1 + f_2}{2}, f_2 \right], U[f_2, B] \right\},
$$

then $U \left( \frac{s_1 + p}{2}, p \right) \leq U(\frac{s_1 + f_2}{2}, f_2)$, and moving $f_2$ to $p$ does not change $\min_{x \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\})$. Otherwise, the maximum above is attained at $U \left[ \frac{s_1 + f_2}{2}, f_2 \right]$, in which case

$$
\min_{x \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, x\}) = N - U \left( \frac{f_1 + s_1}{2}, f_2 \right) = N - U \left( \frac{f_1 + s_1}{2}, p \right),
$$

that is, the payoff is again unchanged, when $f_2$ moves to $p$. □
Remark 3.1. The above proposition implies that given the sorted order of the users in $U$ along $L$, and the placements of the players in round 1, the optimal placements of $P_2$ and $P_1$ in round 2 can be obtained in $O(N)$ and $O(N^2)$ time, respectively. To see this note that computing $\theta(s_2)$ at any point $s_2 \in L$ can be done in $O(N)$ time, and since $\theta(\cdot)$ has $O(1)$ pieces, the optimal payoff of $P_2$ can be found in $O(N)$ time. For the optimal strategy of $P_1$ in round 2, one has to compute $\eta(f_2)$ at $O(N)$ points (by Proposition 3.1 above). As it takes $O(N)$ time to compute $\eta(f_2)$ at single point, the optimal optimal strategy of $P_1$ in round 2 can be computed in $O(N^2)$ time.

3.2 Round 1

The payoff landscapes become increasingly complicated as the number of rounds increases. Here, we compute the landscape of the payoffs in round 1 of the 2-round game: Given a placement $f_1$ by $P_1$ in round 1, the landscape of the payoff of $P_2$ in round 1 is the function \( \psi : L \to \mathbb{R} \):

\[
\psi(s_1) := \min_{f_2 \in L} \max_{s_2 \in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\}),
\]

(3.2)

which is the payoff of $P_2$ when it places the first facility at $s_1$, and both $P_1$ and $P_2$ place their respective facilities optimally in the second round. Similarly, the landscape of the payoff of $P_1$ in round 1 is

\[
\varphi(f_1) := \min_{s_1 \in L} \max_{f_2 \in L} \min_{s_2 \in L} \mathcal{P}_1(\{f_1, f_2\}, \{s_1, s_2\}).
\]

(3.3)

Theorem 3.2. The landscapes of the payoffs of $P_2$ and $P_1$ in round 1 have complexities $O(N^3)$ and $O(N^4)$, respectively.

3.2.1 Proof of Theorem 3.2: Player 2 in Round 1

Given the placement of $f_1$ by $P_1$, $P_2$ can place its first facility either in $[A, f_1)$ or $(f_1, B]$. Here, we analyze the case where $P_2$ places its first facility in the interval $(f_1, B]$. The other case can be done similarly. After $P_2$ places at $s_1$ in round 1, in round 2 $P_1$ can place its facility in either one of the three intervals $[A, f_1]$, $[f_1, s_1]$ or $[s_1, B]$. Therefore, the minimum in (3.2) can be written as:

\[
\psi(s_1) := \min \{\psi_1(s_1), \psi_2(s_1), \psi_3(s_1)\},
\]

where $\psi_1(s_1) := \min_{f_2 \in [A, f_1]} \max_{s_2 \in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\})$, \( \psi_2(s_1) := \min_{f_2 \in [f_1, s_1]} \max_{s_2 \in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\}) \), and \( \psi_3(s_1) := \min_{f_2 \in [s_1, B]} \max_{s_2 \in L} \mathcal{P}_2(\{f_1, f_2\}, \{s_1, s_2\}) \). The proof of Theorem 3.2 for $P_2$ is an immediate consequence of the lemma below.

Lemma 3.1. The functions $\psi_1, \psi_2, \psi_3 : [f_1, B) \to \mathbb{Z}_+ \cup \{0\}$ are piecewise constant, with $O(N)$, $O(N^3)$, and $O(N^2)$ pieces, respectively.

Proof. We begin with $\psi_1$. In this case (referring to Figure 3(b)),

\[
\psi_1(s_1) = U[s_1, B] + U\left(\frac{f_1 + s_1}{2}, s_1\right) + \min_{f_2 \in [A, f_1]} \max \left\{U[\mathcal{A}_1], \mathcal{B}_1 = [f_1, B] \cap U\right\}.
\]

Therefore, the set of points where $\psi_1(\cdot)$ changes its values is $\mathcal{A}_1 \cup \mathcal{B}_1$, where $\mathcal{A}_1 = \{2u_0 - f_1 : u_0 \in [f_1, B] \cap U\} \cap L$ and $\mathcal{B}_1 = [f_1, B] \cap U$. To see this note that if $s_1 \in \mathcal{A}_1$, then $\frac{f_1 + s_1}{2} \in U,$
and therefore $U[f_1, \frac{f_1 + s_1}{2}]$ and, hence $\psi_1(s_1)$, changes in the neighborhood of this point. Similarly, $U[s_1, B]$, and, hence $\psi_1(s_1)$, changes in the neighborhood of $s_1 \in B_1$. As, $|A_1 \cup B_1| = O(N)$, the function $\psi_1$ can have at most $O(N)$ pieces.

Next, we look at $\psi_2$. In this case (referring to Figure 3(a)),

$$\psi_2(s_1) = \min_{f_2 \in [f_1, s_1]} \left\{ U\left( \frac{f_2 + s_1}{2}, B \right) + \max \left\{ U[A, f_1], cov(f_1, f_2), U\left( f_2, \frac{f_2 + s_1}{2} \right) \right\} \right\}.$$ 

Let $\hat{f}_2 \in [f_1, s_1]$ be a point which attains the minimum above. Then as $s_1$ varies over $L$, $\psi_2(s_1)$ changes when $\hat{f}_2 + s_1 \in U$. We now argue that it suffices to assume (a) $\hat{f}_2 \in U$, or (b) $\hat{f}_2 = f_1 + 2|u_a - u_b|$, for some $u_a, u_b \in U$. Then there are two cases:

- Let $f'_2 < \hat{f}_2$ be such that $cov(f_1, f'_2) = U[u_a, u_b]$ and $f'_2 = f_1 + 2|u_a - u_b|$, where $u_a, u_b \in U$ is such that $cov(f_1, f'_2) = U[u_a, u_b]$. If $U[f'_2, f_2]$ is empty, then the value of $\psi_2(s_1)$ remains unchanged, when $\hat{f}_2$ is replaced by $f'_2$, that is, (b) holds.

- Otherwise, move $\hat{f}_2$ to the closest user $u'$ to the left of $\hat{f}_2$ in $[f'_2, \hat{f}_2]$. Then $cov(f_1, \hat{f}_2) = cov(f_1, u')$, and, therefore, $\psi_2(s_1)$ remain unchanged, that is, (a) holds in this case.

Therefore, the set of points where $\psi_2(\cdot)$ changes is contained in $A_2 \cup B_2$, where $A_2 := \bigcup_{u \in U} \{2u - f : f = f_1 + 2|u_a - u_b| \text{ for some } u_a, u_b \in U \}$ and $B_2 = \bigcup_{u \in U} \{2u - f : f \in U \}$. The result about $\psi_2(\cdot)$ follows by noting that $|A_2 \cup B_2| = O(N^3)$, as required.

Finally, we consider $\psi_3(s_1)$. Again, referring to Figure 3(c), it follows that

$$\psi_3(s_1) = \min_{f_2 \in [s_1, B]} \left\{ U\left( \frac{f_1 + s_1}{2}, \frac{s_1 + f_2}{2} \right) + \max \left\{ U[A, f_1], U\left( f_1, \frac{f_1 + s_1}{2} \right), U\left( \frac{s_1 + f_2}{2}, f_2 \right), U(f_2, B) \right\} \right\}.$$ 

This can change, for points in the neighborhood of $s_1$ such that either (a) $\frac{s_1 + f_1}{2} \in U$ or (b) $\frac{s_1 + f_2}{2} \in U$. The set of points where (a) happens is $A_3 := \{2u - f_1 : u \in [f_1, B) \cap U \}$. For (b) note that the optimal placement of $f_2$ can be obtained by checking in the neighborhood of users in $[s_1, B]$ (recall the third case in the proof of Proposition 3.1). Therefore, the set of points where $s_1$ can be placed such that if $f_2 \in [s_1, B) \cap U$, then $\frac{s_1 + f_2}{2} \in U$, is contained in $B_3 := \{2u_a - u_b : u_a, u_b \in \{f_1, B) \cap U \}$. This implies that the function $\psi_3(s_1)$ can have at most $O(N^2)$ pieces, as $|A_3 \cup B_3| = O(N^2)$.

### 3.2.2 Player 1 in Round 1

Recall the definition of the landscape of the payoff of P1 in round 1 from (3.3). Then

$$\varphi(f_1) = \min_{s_1 \in L} \min_{f_2 \in L} \min_{s_2 \in L} P_1(\{f_1, f_2\}, \{s_1, s_2\}) = \min\{\varphi_1(f_1), \varphi_2(f_1)\},$$

where $\varphi_1(f_1) := \min_{s_1 \in [A, f_1]} \max_{f_2 \in L} \min_{s_2 \in L} P_1(\{f_1, f_2\}, \{s_1, s_2\})$, that is, P2 is restricted to place in the interval $[A, f_1]$ in round 1, and, $\varphi_2(f_1) := \min_{s_1 \in [f_1, B]} \max_{f_2 \in L} \min_{s_2 \in L} P_1(\{f_1, f_2\}, \{s_1, s_2\})$, where P2 is restricted to place in the interval $[f_1, B]$ in round 1. Both $\varphi_1(f_1)$ and $\varphi_2(f_1)$ are piecewise constant functions as $f_1$ varies in $L$. In the following, we will describe the complexity of the graph of $\varphi_2(f_1)$, and can be $\varphi_1(f_1)$ done similarly.

---

4 Given $x, y \in L$, there exists $u_a, u_b \in U \cap [x, y]$ such that $cov(x, y) = U[u_a, u_b]$, because any interval $I \subset [x, y]$ which attains $cov(x, y) = |U \cap I|$ can be shrunk to an interval $[u_a, u_b]$ such that $cov(x, y) = U[u_a, u_b]$. 

---

10
Observation 3.1. Let $f_1, s_1$ be placement of facilities by P1 and P2 in round 1, such that $U[f_1, s_1] > 0$. Now, if the optimal placement of P1 in round 2 is at $f_2 \in (A, f_1)$, then it is possible to move $s_1$ immediately to the right of an user in $U$, without decreasing the payoff of P2.

Proof. Let $u'_1 < u'_2 < \ldots < u'_N$ be the sorted order of the users in $U$, such that $s_1 \in (u'_j, u'_{j+1})$. If we move $s_1$ immediately to the right of $u'_j$ (which we denote by $s'_1$), then $U[s_1, B] = U[s'_1, B]$, but the point $\frac{f_1 + s_1}{2}$ moves left to $\frac{f_1 + s'_1}{2}$, which may lead to $f_2$ to move to a point $f'_2$ in $(f_1, s'_1)$. Then the payoff of P2 in round 2, when $f_2$ moves to $f'_2$ (after $s_1$ moves to $s'_1$) is

$$U \left( \frac{f'_2 + s'_1}{2}, B \right) + \max \left\{ U[A, f_1], \text{cov}(f_1, f'_2), U \left[ f'_2, \frac{f'_2 + s'_1}{2} \right] \right\}$$

$$\geq U \left( \frac{f'_2 + s_1}{2}, B \right) + \max \left\{ U[A, f_1], \text{cov}(f_1, f'_2), U \left[ f'_2, \frac{f'_2 + s_1}{2} \right] \right\}$$

$$\geq U \left( \frac{f_1 + s_1}{2}, B \right) + \max \left\{ U[A, f_2], \text{cov}(f_1, f_2), U \left[ f_1, \frac{f_1 + s_1}{2} \right] \right\},$$

where the last inequality uses the assumption that $f_2$ is the optimal placement of P1 in round 2 (hence the payoff of P2 when P1 places at $f'_2$ instead of at $f_2$ will be larger.) This shows that the optimal location of P1 in round 2 remains unchanged when $s_1$ moves to $s'_1$, completing the proof of the lemma.

\[ \text{Figure 4: Optimal strategy of P1 in round 1: (a) Case 1, (b) Case 2, (c) Case 3.} \]

Now, we compute the complexity of $\varphi_2(\cdot)$. Recall, we are assuming $s_1 \in (f_1, B]$, and depending on the location of $f_2$ there are 3 cases:

**Case 1:** $f_2 < f_1 < s_1$. In this case, the payoff of P1 is

$$N - U \left[ \frac{f_1 + s_1}{2}, B \right] - \max \left\{ U[A, f_2], \text{cov}([f_2, f_1]), U \left[ \frac{f_1 + s_1}{2}, B \right] \right\}.$$

In this case, as $f_1$ moves along $L$, the payoff above changes when either $\text{cov}([f_2, f_1])$ changes, or $\frac{f_1 + s_1}{2}$ or $f_2$ passes through an user. Note that, in this case, we can essentially assume $s_1 \in U$ (by Observation 3.1 it suffices to check immediately to left or right of users). Therefore, the set of points $f_1$ for which $U[f_1, \frac{f_1 + s_1}{2}]$ changes can be expressed as $A_1 = \{ 2u_a - u_b : u_a, u_b \in U \}$. Next, we try to find the set of $f_1$ for which $\text{cov}([f_2, f_1])$ changes. Note that for each possible value of $\text{cov}([f_2, f_1])$, there is an interval $[u_a, u_b]$, where $u_a, u_b \in U$, such that $\text{cov}([f_2, f_1]) = U[u_a, u_b]$. Moreover, it suffices to assume that in round 2, P1 places $f_2 \in U$ (recall the second case in the proof of Proposition 3.1). Therefore, considering each possible placement of $f_2$ on an user and all pair of users in $U$, the set of points $f_1$ for which $\text{cov}([f_2, f_1])$ changes is contained in $B_1 = \{ u_a \pm 2|u_b - u_c| : u_a, u_b, u_c \in U \}$. As $|A_1 \cup B_1| = O(N^2)$, the proof of this case is complete.
Remark 3.2. By Theorem 3.2, for the optimal strategy of P2 in round 1, it suffices to compute \( O \psi \) found in \( \phi \) of P2 in round 1. For the optimal strategy of P1 in round 1, Theorem 3.2 implies that it is enough to compute \( O f \) values of \( f \) as before, for each possible value of \( f \) such that \( \frac{f_1 + s_1}{2} = \frac{f_2 + s_1}{2} \) there exists \( u_a, u_b \in U \) such that \( U(\frac{f_1 + s_1}{2}, \frac{f_2 + s_1}{2}) = U[u_a, u_b] \). Therefore, given a placement \( f_2 \in U \), we can move \( s_1 \) either to an user or to a point such that \( \frac{f_1 + s_1}{2} \in U \) keeping the payoff of P1 unchanged, and given the location of \( s_1 \) we can move \( f_1 \) either to an user or to a point such that \( \frac{s_1 + f_2}{2} \in U \) (again keeping the payoff of P1 unchanged). This means the set of possible values of \( s_1 \) is contained in \( B_2 := \{2u_a - u_b : u_a, u_b \in U\} \) (since \( f_2 \in U \)), and the set of possible values of \( f_1 \) is contained in \( A_2 := \{2u_c - u' : u_c \in U \text{ and } u' \in B_2\} \), which satisfies \( |A_2| = O(N^3) \). (Note that both the sets \( A_2 \) and \( B_2 \) contain the user set \( U \).)

Case 2: \( f_1 < s_1 < f_2 \). In this case, the payoff of P1 is

\[
N - U\left( \frac{f_1 + s_1}{2}, \frac{f_2 + s_1}{2} \right) - \max\left\{ U[A, f_1], U\left( \frac{f_1 + s_1}{2}, f_2 \right), U\left( \frac{s_1 + f_2}{2}, B \right) \right\}.
\]

As before, for each possible value of \( \text{cov}([f_2, f_1]) \), there is an interval \([u_a, u_b]\), where \( u_a, u_b \in U \), such that \( \text{cov}([f_2, f_1]) = U[u_a, u_b] \). As \( f_1 \) moves along \( L \), the payoff above will change if one of the following 3 cases happen:

- \( f_1 \) coincides with an user: In this case, the set of possible choices of \( f_1 \) is just \( U \).
- \( f_2 \) coincides with an user: In this case, the set of possible choices of \( f_1 \) is contained in \( A_{31} := \{u_c - 2|u_a - u_b|, \text{ for } u_a, u_b, u_c \in U\} \), using \( f_2 - f_1 = 2|u_a - u_b| \) and \( f_2 \in U \).
- \( \frac{f_2 + s_1}{2} \) coincides with a user: In this case, we can move \( s_1 \) to the closest user to its left and \( f_2 \) to its right so the midpoint \( \frac{f_2 + s_1}{2} \) remain unchanged (otherwise, one of the previous two cases happen), without changing the payoff of P1. As \( s_1 \) and \( \frac{f_2 + s_1}{2} \) now both coincide with users, this gives \( B'_3 = \{2u_c - u_d : u_c, u_d \in U\} \) choices for \( f_2 \), and \( A_{32} = \{f - 2|u_a - u_b| : f \in B'_3 \text{ and } u_a, u_b \in U\} \) choices for \( f_1 \).

Therefore, the set of possible choices of \( f_1 \) is contained in \( U \cup A_{31} \cup A_{32} \), with \( |U \cup A_{31} \cup A_{32}| = O(N^4) \), completing the proof. \( \square \)

Remark 3.2. By Theorem 3.2, for the optimal strategy of P2 in round 1, it suffices to compute \( \psi(s_1) \) at \( O(N^3) \) points. As computing the optimal payoff of P2 at a point in round 1 requires \( O(N^2) \) time (Remark 3.1), this gives an \( O(N^5) \) time algorithm for computing the optimal strategy of P2 in round 1. For the optimal strategy of P1 in round 1, Theorem 3.2 implies that it is enough to compute \( \varphi(f_1) \) in \( O(N^4) \) points, which implies the optimal strategy of P1 in round 1 can be found in \( O(N^9) \) time.

References


