Geometric Proof of a Ramsey-Type Result For Disjoint Empty Convex Polygons I

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Abstract

This is the first part of a two part paper where we give a geometric proof of a Ramsey-type result for disjoint empty convex polygons. We prove that every set of 11 points in the plane, no three on a line, contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. This result was established by Aichholzer et al. [1] with the help of the order type data base [2, 3]. In this two part paper we give a geometric proof of this fact which requires only a moderate number of case distinctions. In this part of the paper we prove some basic observations, following which we show that any set of 11 points in the plane, no three on a line, with (a) at most six points in the convex hull and exactly one point in the third convex layer, or (b) at least six points in the convex hull, contains an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral.

Keywords. Discrete geometry, Convex polygons, Erdős-Szekeres theorem, Ramsey-type results.

1 Introduction

The origin of the problems concerning the existence of convex empty polygons in planar point sets goes back to the famous theorem due to Erdős and Szekeres [7]. It states that for every positive integer $m \geq 3$, there exits a smallest integer $ES(m)$, such that any set of n points ($n \geq ES(m)$) in the plane, no three on a line, contains a subset of m points which lie on the vertices of a convex polygon. Evaluating the exact value of $ES(m)$ is a long standing open problem. A construction due to Erdős [8] shows that $ES(m) \geq 2^{m-2}+1$, which is also conjectured to be sharp. It is known that $ES(4) = 5$ and $ES(5) = 9$ [15]. Following a long computer search, Szekeres and Peters [18] recently proved that $ES(6) = 17$. The value of $ES(m)$ is unknown for all $m > 6$. The best known upper bound for $m \geq 7$ due to Toth and Valtr [19] is $ES(m) \leq \binom{2m-5}{m-3} + 1$.

In 1978 Erdős [6] asked whether for every positive integer k, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains k points which lie on the vertices of convex polygon whose interior contains no points of the set. Such a subset is called an *empty convex k-gon* or a *k-hole*. Esther Klein showed $H(4) = 5$ and Harborth [10] proved that $H(5) = 10$. Horton [12] showed that it is possible to construct arbitrarily large set of points without a 7-hole, thereby proving that $H(k)$ does not exist for $k \geq 7$. The existence of 6-holes remained open for a long time, until recently, Gerken [9] and independently Nicolás [17] proved its existence. Later, Valtr [21] gave a simpler version of Gerken's proof. The exact value of $H(6)$ is, however, still unknown.

The problems concerning the existence of disjoint holes, that is, empty convex polygons with non-intersecting convex hulls, was first studied by Urabe [20] while addressing the problem of partitioning of planar point sets. Urabe [20] proved that every set of 7 points can be partitioned into a triangle and a disjoint convex quadrilateral. Hosono and Urabe [14] showed that every set of 9 points contains two disjoint 4-holes. Later, they proved that every set of 10 points contains a 5-hole and a disjoint 3-hole and every set of 14 points contains a 5-hole and a disjoint 4-hole [13]. Some of these results were later reconfirmed by Wu and Ding [22]. For results regarding the number of holes in planar point sets and other related problems and conjectures see the surveys by Bárány and Károlyi [4] and Morris and Soltan [16].

Over the years, it has been realized that small point sets are, in general, notoriously difficult to handle. In fact, the asymptotic lower bound for the number of order types of a set of n points in the plane is $n^{\Theta(n \log n)}$ [11]. This has prompted researchers to the computer aided order type enumeration method, developed few years ago by Aichholzer et al. [2, 3], for proving such results for small point sets. While addressing the problem of pseudo-convex decomposition, Aichholzer et al. [1] proved the following two theorems using the order type data base [2, 3].

Theorem 1 *Every set of 8 points in the plane, no three on a line, either contains a 5-hole or two disjoint 4-holes.*

Theorem 2 *Every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole.*

Their paper contains a simple geometric proof of Theorem 1 based on a few case distinctions. However, Theorem 2 is only verified using the order type data base [2, 3]. In this two part paper, we give an intuitive geometric proof of Theorem 2 which requires the careful analysis of a moderate number of case distinctions. We begin the first part with some basic observations. Using them we show that any set of 11 points in the plane, no three on a line, with at most six points in the convex hull and exactly one point in the third convex layer contains an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. Finally, we show that any set of 11 points in the plane with at least six points in the convex hull also contains an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. The remaining cases are proved in the next part of the paper.

Aichholzer et al. [1] used the above theorems to establish bounds on the pseudo-convex decomposition number of planar point sets. Recently, using Theorem 2, we proved that the minimum number of points in the plane required to obtain a 5-hole and a disjoint 4-hole is 12 [5], which answers a question of Hosono and Urabe [13]. The minimum number of points in the plane required to obtain two disjoint 5-holes still remains open [14].

2 Notations and Definitions

We first introduce the definitions and notations required for the remainder of the paper. Let S be a finite set of points in the plane in general position, that is, no three on a line. Denote the convex hull of S by $CH(S)$. The boundary vertices of $CH(S)$, and the points of S in the interior of $CH(S)$ are denoted by $\mathcal{V}(CH(S))$ and $\mathcal{I}(CH(S))$, respectively. A region R in the plane is said to be empty in S if R contains no elements of S in the interior. Moreover, for any set T, $|T|$ denotes the cardinality of T.

By $P := p_1p_2\dots p_k$ we denote a convex k-gon with vertices $\{p_1, p_2, \dots, p_k\}$ taken in anti-clockwise order. $\mathcal{V}(P)$ denotes the set of vertices of P and $\mathcal{I}(P)$ the interior of P. The collection of all points $q \in \mathbb{R}^2$ such that $\{q\} \cup \mathcal{V}(P)$ form a convex $(k+1)$-gon is called the forbidden zone of P.

The *j-th convex layer* of S, denoted by $L\{j, S\}$, is the set of points S that lies on the boundary of $CH(S\backslash\{\bigcup_{i=1}^{j-1} L\{i, S\}\})$, where $L\{1, S\} = \mathcal{V}(CH(S))$. $|L\{j, S\}|$ denotes the number of points of S in j-th convex layer and $\mathcal{I}(L\{j, S\})$ is the set of points in S which lies in the interior of $CH(L\{j, S\})$. If $p,\ q \in S$ be such that pq is an edge of the convex hull of the j-th layer, then the open halfplane bounded by the line pq and not containing any point of $S\backslash\{\bigcup_{i=1}^{j-1} L\{i, S\}\}$ will be referred to as the *outer* halfplane induced by the edge pq. For any three points $p, q, r \in S$, $\mathcal{H}(pq, r)$ (respectively $\mathcal{H}_c(pq, r)$) denotes the open (respectively closed) halfplane bounded by the line pq containing the point

r. Similarly, $\overline{\mathcal{H}}(pq,r)$ (respectively $\overline{\mathcal{H}}_c(pq,r)$) is the open (respectively closed) halfplane bounded by pq not containing the point r. Moreover, if $\angle rpq < \pi$, $Cone(rpq)$ denotes the interior of the angular domain $\angle rpq$.

3 Towards the Proof of Theorem 2

We begin the proof of the theorem with a simple observation about the existence of 5-holes.

Observation 1 *Let Z be a set of points in the plane in general position, with $|CH(Z)| = 5$ and $|\mathcal{I}(CH(Z))| = 2$. Then Z contains a 5-hole.*

Proof. The two points $x, y \in \mathcal{I}(CH(Z))$ span a line which divides the plane into two halfplanes, one of which must contain at least three points of $\mathcal{V}(CH(S))$. These three points along with the points x and y form a 5-hole. □

Next, we prove a sufficient condition for a set of 10 points in the plane, in general position, to contain a 6-hole or a 5-hole and a disjoint 4-hole.

Observation 2 *Any set Z of 10 points in general position, with $|CH(Z)| \geq 7$, or $|CH(Z)| = 6$ and $|L\{2, Z\}| = 4$, contains a 6-hole or a 5-hole and a disjoint 4-hole.*

Proof. To begin with, suppose $|CH(Z)| = 6$ and $|L\{2, Z\}| = 4$. Consider the partition of the exterior of $L\{2, Z\}$ into disjoint regions R_i as shown in Figure 1(a). Now, Z contains a 6-hole unless $|R_1| + |R_5| \leq 1$. Moreover, if $|R_2| + |R_3| + |R_4| + |R_6| + |R_7| + |R_8| \geq 5$, then by the pigeonhole principle either $|R_2| + |R_3| + |R_4| \geq 3$ or $|R_6| + |R_7| + |R_8| \geq 3$. Without loss of generality, let $|R_2|+|R_3|+|R_4| \geq 3$. Now, $\overline{\mathcal{H}}_c(p_1p_2, p_3)$ forms a 6-hole whenever $|R_2| + |R_3| + |R_4| \geq 4$. Moreover, if $|R_2| + |R_3| + |R_4| = 3$, then $\overline{\mathcal{H}}_c(p_1p_2, p_3)$ forms a 5-hole which is disjoint from the 4-hole formed by $\mathcal{H}(p_1p_2, p_3)$, since $H(4) = 5$. Therefore, $|R_2| + |R_3| + |R_4| + |R_6| + |R_7| + |R_8| \leq 4$, which implies that $\sum_{i=1}^{8} |R_i| \leq 5$, which contradicts $|CH(Z)| = 6$.

Now, suppose $|CH(Z)| = 7$ with $p, q, r \in L\{2, Z\}$. Then there exists an edge of pqr, say pq, such that $|\overline{\mathcal{H}}(pq, r) \cap Z| \geq 3$. If $|\overline{\mathcal{H}}(pq, r) \cap Z| \geq 4$, then $\overline{\mathcal{H}}_c(pq, r) \cap Z$ forms a 6-hole. Otherwise, $|\overline{\mathcal{H}}(pq, r) \cap Z| = 3$, and the 5-hole formed by $\overline{\mathcal{H}}_c(pq, r) \cap Z$ is disjoint from the 4-hole contained in $\mathcal{H}(pq, r) \cap Z$, since $|\mathcal{H}(pq, r) \cap Z| = 5$.

Finally, it is easy to see that Z contains a 6-hole whenever $|CH(Z)| \geq 8$. □

We say that a set of 11 points is *admissible* if it contains either a 6-hole or a 5-hole and a 4-hole which are disjoint. Let S be a set 11 points in general position. Let $|L\{1, S\}| = |CH(S)| = k$ and $|L\{2, S\}| = |CH(\mathcal{I}(CH(S)))| = m$

with $\mathcal{V}(CH(S)) = L\{1,S\} = \{s_1, s_2, \ldots, s_k\}$ and $L\{2,S\} = \{p_1, p_2, \ldots, p_m\}$, where the vertices are taken in counter-clockwise order. While indexing a set of points from $L\{1,S\}$ or $L\{2,S\}$, we identify indices modulo k or modulo m, respectively.

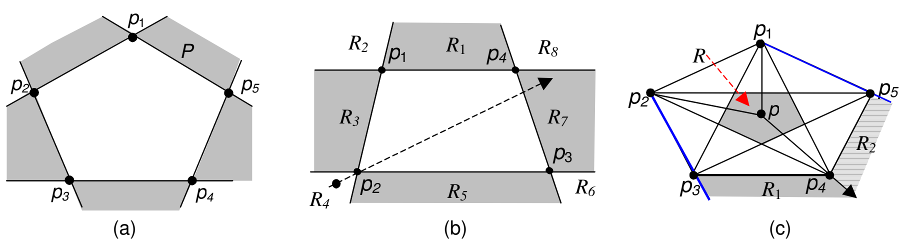

Figure 1: (a) Forbidden zone of a pentagon P, Illustration of the proof of Lemma 1: (b) $|L\{2,S\}| = 4$, (c) $|L\{2,S\}| = 5$.

Next, we have the following simple observation:

Observation 3 *If there are more than two points of $\mathcal{V}(CH(S))$ in some outer halfplane induced by an edge of $CH(L\{2,S\})$, then S is admissible.*

Proof. If some outer halfplane induced by an edge of $L\{2,S\}$ contains more than three points of S, then S contains a 6-hole. Otherwise, let $p,q,r \in L\{2,S\}$ be such that pq is an edge of $CH(L\{2,S\})$, and $|\overline{\mathcal{H}}(pq,r) \cap S| = 3$. Then the 5-hole formed by $\overline{\mathcal{H}}_c(pq,r) \cap S$ is disjoint from the 4-hole contained in $\mathcal{H}(pq,r) \cap S$, since $|\mathcal{H}(pq,r) \cap S| \geq 5$. □

We now present the first main result of this paper, which is a crucial part in the proof of Theorem 2.

Lemma 1 *Any set S of 11 points in general position, with $|CH(S)| \leq 6$ and $|L\{3,S\}| = 1$, is admissible.*

Proof. Let $p \in L\{3,S\}$. Since $3 \leq |CH(S)| \leq 6$, we have $4 \leq |L\{2,S\}| \leq 7$. We consider the different cases based on size of $|L\{2,S\}|$.

$|L\{2,S\}| = 4$: Consider the partition of the exterior of $L\{2,S\}$ into disjoint regions R_i as shown in Figure 1(a). If for some point $p_i \in L\{2,S\}$ we have $|\overline{\mathcal{H}}(p_ip_{i+1}, p_{i+2}) \cap S| = 1$, then $|CH(\mathcal{H}_c(p_ip_{i+1}, p_{i+2}) \cap S)| \geq 7$, since $|CH(S)| \geq 6$. The admissibility of S then follows from Observation 2. Moreover, from Observation 3, S is admissible whenever $|\overline{\mathcal{H}}(p_ip_{i+1}, p_{i+2}) \cap S| \geq 3$. Therefore, for all the points $p_i \in L\{2,S\}$, $|\overline{\mathcal{H}}(p_ip_{i+1}, p_{i+2}) \cap S| = 2$, that is, every outer halfplane induced by the edges of $CH(L\{2,S\})$ contains exactly two points of $\mathcal{V}(CH(S))$. This means,

$$\begin{aligned} |R_2| + |R_3| + |R_4| = 2, \\ |R_4| + |R_5| + |R_6| = 2, \\ |R_6| + |R_7| + |R_8| = 2, \\ |R_8| + |R_1| + |R_2| = 2. \end{aligned} \tag{1}$$

Adding these equations and using the fact $\sum_{i=1}^{8} |R_i| = |\mathcal{V}(CH(S))| = 6$, we get $|R_2| + |R_4| + |R_6| + |R_8| = 2$ and so $|R_1| + |R_3| + |R_5| + |R_7| = 4$. Now, if $|R_1| \geq 2$, then $(R_1 \cap S) \cup \{p_1, p_4, p\}$ forms a 5-hole which is disjoint from the 4-hole in $\mathcal{H}_c(p_2p_3, p)$. Therefore, $|R_1| \leq 1$, and similarly $|R_3|, |R_5|, |R_7| \leq 1$. This implies, $|R_1| = |R_3| = |R_5| = |R_7| = 1$. Without loss of generality, we can now assume, $|R_4| = |R_8| = 1$. Depending upon the location of p, either $\overline{\mathcal{H}}_c(p_1p_2, p) \cup \{p\}$ or $\overline{\mathcal{H}}_c(p_2p_3, p) \cup \{p\}$ forms a 5-hole, which is disjoint from the 4-hole formed by $\overline{\mathcal{H}}_c(p_3p_4, p)$ or $\overline{\mathcal{H}}_c(p_1p_4, p)$, respectively.

$|L\{2,S\}| = 5$: To begin with, let $|\overline{\mathcal{H}}(p_3p_4, p_5) \cap S| = 1$, then $|CH(\mathcal{H}_c(p_3p_4, p_5) \cap S)| \geq 6$. Let $Z = \mathcal{H}_c(p_3p_4, p_5) \cap S$. Clearly, $|Z| = 10$ and $|CH(Z)| \geq 6$. If $|CH(Z)| \geq 7$ or $|CH(L\{2, Z\})| = 4$, then S is admissible by Observation 2. Therefore, let $|CH(\mathcal{I}(Z))| = 3$ which means $p \in \mathcal{I}(p_1p_2p_5)$ (Figure 1(b)). Now, Z contains a 6-hole unless, $|Cone(p_1pp_2) \cap Z| = |Cone(p_2pp_5) \cap Z| = |Cone(p_1pp_5) \cap Z\}| = 2$. Now, by the pigeonhole principle either $|\mathcal{H}(pp_1, p_2) \cap S| \geq 5$ or $|\mathcal{H}(pp_1, p_5) \cap S| \geq 5$. Without loss of generality, let $|\mathcal{H}(pp_1, p_2) \cap S| \geq 5$. Then the 4-hole contained in $\mathcal{H}(pp_1, p_2) \cap S$ is disjoint from the 5-hole formed by $Cone(p_1pp_5) \cap S \cup \{p_1, p_5, p\}$. Therefore, assume that $|\overline{\mathcal{H}}(p_3p_4, p_5) \cap S| \geq 2$. Observation 3 now implies that S is admissible unless $|\overline{\mathcal{H}}(p_3p_4, p_5) \cap S| = 2$. Since this should be true for all the edges of $L\{2,S\}$, it follows that every outer halfplane induced by the edges of $L\{2,S\}$ contains exactly two point of $\mathcal{V}(CH(S))$. Let R be the shaded region inside $L\{2,S\}$ as shown in Figure 1(c). Consider following two subcases:

- **$p \notin R$:** Clearly, S is admissible unless the forbidden zone of the 5-hole $pp_2p_3p_4p_5$ is empty in S. If $|Cone(p_1pp_2) \cap L\{1,S\}| \geq 2$, $(Cone(p_1pp_2) \cap L\{1,S\}) \cup \{p_1, p_2, p\}$ forms a 5-hole which is disjoint from the 4-hole contained in $\overline{\mathcal{H}}_c(p_3p_4, p) \cap S$, since $|\overline{\mathcal{H}}(p_3p_4, p) \cap S| = 2$. Therefore, assume $|Cone(p_1pp_2) \cap L\{1,S\}| \leq 1$, and similarly $|Cone(p_1pp_5) \cap L\{1,S\}| \leq 1$. Moreover, since $|\overline{\mathcal{H}}(p_3p_4, p_1) \cap S| = 2$, this implies that $|\mathcal{V}(CH(S))| < 4$, which is a contradiction.

- **$p \in R$:** Let the regions R_1 and R_2 be as shown in Figure 1(c). Without loss of generality, it can be assumed that $R_1 \cup R_2$ is non-empty in

S. Let $s_i \in (R_1 \cup R_2) \cap S$. If $s_i \in R_1 \cap S$, $p_2pp_4s_ip_3$ is a 5-hole which is disjoint from the 4-hole formed by $\overline{\mathcal{H}}_c(p_1p_5, p)$. Similarly, for $s_i \in R_2 \cap S$, $p_1pp_4s_ip_5$ is a 5-hole, which is disjoint from the 4-hole formed by $\overline{\mathcal{H}}_c(p_2p_3, p)$.

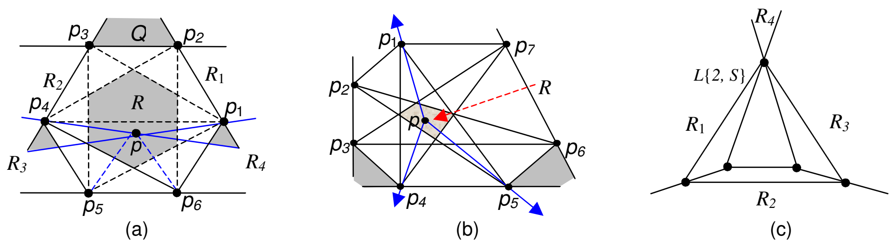

Figure 2: Illustration of the proof of Lemma 1: (a) $|L\{2, S\}| = 6$, (b) $|L\{2, S\}| = 7$, (c) Illustration of the proof of Lemma 2 when $|L\{3, S\}| = 2$.

$|L\{2, S\}| = 6$: Refer to Figure 2(a). Clearly S contains a 6-hole whenever $p \notin R$. Therefore, let $p \in R$ and assume that $p \in \mathcal{I}(p_1p_4p_5p_6)$. Consider the partition of the exterior of $CH(L\{2, S\})$ into disjoint regions R_i as shown in Figure 2(a). Observe that S contains a 6-hole unless $|R_1| = |R_2| = 0$, and $|R_3|, |R_4| \leq 1$. Also, if $|\overline{\mathcal{H}}(p_5p_6, p_1) \cap S| \geq 2$, the 5-hole $p_1p_2p_3p_4p$ is disjoint from the 4-hole contained in $\overline{\mathcal{H}}_c(p_5p_6, p_1) \cap S$. Therefore, let $|\overline{\mathcal{H}}(p_5p_6, p_1) \cap S| = 1$. Now, if $|\overline{\mathcal{H}}(p_2p_3, p_1) \cap S| = 2$, then $|R_3| + |R_4| = 1$, since $|CH(S)| = 4$. Then the 4-hole formed by $\overline{\mathcal{H}}_c(p_2p_3, p_1) \cap S$ is disjoint from the 5-hole contained in either $(R_3 \cap S) \cup \{p_4, p_5, p_6, p\}$ or $(R_4 \cap S) \cup \{p_1, p_6, p_5, p\}$. So, assume $|\overline{\mathcal{H}}(p_2p_3, p_1) \cap S| = 1$ and $|R_3| = |R_4| = 1$. Again, S is admissible whenever $|\mathcal{H}(p_4p, p_6) \cap S| \geq 5$ or $|\mathcal{H}(p_1p, p_6) \cap S| \geq 5$. Otherwise, both $|(R_3 \backslash \mathcal{H}(p_1p, p_6)) \cap S| \geq 1$ and $|(R_4 \backslash \mathcal{H}(p_4p, p_6)) \cap S| \geq 1$. Observe that $p_1p_2p_3p_4p$ can be extended to a 6-hole if Q is non-empty in S. So, assume that Q is empty in S, which implies that $|\overline{\mathcal{H}}(p_1p_2, p_3) \cap S| = 2$ or $|\overline{\mathcal{H}}(p_3p_4, p_1) \cap S| = 2$ (see Figure 2(a)). Now, if $|\overline{\mathcal{H}}(p_1p_2, p_3) \cap S| = 2$, $(R_3 \cap S) \cup \{p, p_4, p_5, p_6\}$ forms a 5-hole which is disjoint from the 4-hole formed by $\overline{\mathcal{H}}_c(p_1p_2, p_3) \cap S$. Similarly, for the case $|\overline{\mathcal{H}}(p_3p_4, p_1) \cap S| = 2$.

$|L\{2, S\}| = 7$: Let R be the shaded region inside $L\{2, S\}$ as shown in Figure 2(b). If $p \notin R$, S contains a 6-hole. Therefore, assume that $p \in R$. For all $i \in \{1, 2, \ldots, 7\}$, $H_i = p_ip_{i+1}p_{i+2}p_{i+3}p$ is a 5-hole. S contains a 6-hole whenever the forbidden zones of H_i are non-empty in S. Therefore, assume that the forbidden zones of each of H_i are empty in S. This implies that every point in $L\{1, S\}$ must be in the regions where at least three outer halfplanes induced by the edges of $CH(L\{2, S\})$ intersect.

Since there are 7 outer halfplanes induced by the edges of $CH(L\{2, S\})$, the pigeon-hole principle implies, some outer halfplane induced by an edge of $CH(L\{2, S\})$, say $\overline{p_3p_4}$, contains at least $\lceil \frac{3\times 3}{7} \rceil = 2$ points of $L\{1, S\}$. Then the 5-hole $p_1p_2pp_5p_7$ is disjoint from the 4-hole contained in $\overline{\mathcal{H}}_c(p_3p_4, p_1) \cap S$. $\square$

With the help of Lemma 1, we now prove the admissibility of S when $|CH(S)| \geq 6$.

Lemma 2 *S is admissible whenever $|CH(S)| \geq 6$.*

Proof. To begin with let $|CH(S)| \geq 7$. Then $|\mathcal{I}(L\{1, S\}| \leq 4$ and Observation 3 implies that every outer halfplane induced by the edges of $L\{2, S\}$ contains at most two points of $\mathcal{V}(CH(S))$. This is possible only when, $|L\{2, S\}| = 4$ and so $|CH(S)| = 7$. Now, there must exist an edge of $L\{2, S\}$, say p_ip_j, such that $|\overline{\mathcal{H}}(p_ip_j, p_k) \cap S| \geq 2$, where p_k is any other point of $L\{2, S\}$. Observation 1 implies, $\mathcal{H}(p_ip_j, p_k) \cap S$ contains a 5-hole. This 5-hole is disjoint from the 4-hole contained in $\overline{\mathcal{H}}_c(p_ip_j, p_k) \cap S$.

Next, assume that $|CH(S)| = 6$. If $|L\{3, S\}| = 1$, the admissibility follows from Lemma 1. We consider the remaining two cases based on the size of $L\{3, S\}$ as follows:

$|L\{3, S\}| = 2$: This means, $|L\{2, S\}| = 3$. Consider the partition of the exterior of $L\{2, S\}$ into disjoint regions R_i, as shown in Figure 2(c). Observe that S contains a 6-hole unless $|R_2| \leq 1$, $|R_1| + |R_4| \leq 2$, and $|R_3| \leq 2$. This implies $\sum_{i=1}^{4} |R_i| \leq 5 < |\mathcal{V}(CH(S))|$, which is a contradiction.

$|L\{3, S\}| = 0$: Then $L\{2, S\}$ is a 5-hole. If the forbidden zone of $L\{2, S\}$ is non-empty in S, then S contains a 6-hole. Otherwise, every point in $L\{1, S\}$ must lie in the regions where more than one of the outer halfplanes, induced by the edges of $CH(L\{2, S\})$, intersect. Since there are 5 outer halfplanes induced by the edges of $CH(L\{2, S\})$, by the pigeon-hole principle, there must exist some outer halfplane induced by an edge of $CH(L\{2, S\})$ which contains at least $\lceil \frac{6\times 2}{5} \rceil = 3$ points of $\mathcal{V}(CH(S))$. The admissibility of S now follows from Observation 3. $\square$

4 Conclusions

This is the first part of two part paper where we give a geometric proof of a Ramsey-type for disjoint empty convex polygons, which states that every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole. In this part, we prove that S is admissible whenever

$|CH(S)| \leq 6$ and $|L\{3, S\}| = 1$. Using this we show the admissibility of S whenever $|CH(S)| \geq 6$.

In part II, these results will be used to prove the admissibility of S in the remaining cases, that is, when $|CH(S)| \leq 5$ and $|L\{3, S\}| \geq 2$.

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