Geometric Proof of a Ramsey-Type Result For Disjoint Empty Convex Polygons II

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Abstract

This is the second part of a two part paper where we give a geometric proof of a Ramsey-type result for disjoint empty convex polygons. We prove that every set of 11 points in the plane, no three on a line, contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. This result was established by Aichholzer et al. [1] with the help of the order type data base [2, 3]. In the first part we proved that any set of 11 points in the plane, no three on a line, with (a) at most six points in the convex hull and exactly one point in the third convex layer, or (b) at least six points in the convex hull, contains an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. In this part we show the existence of an empty convex hexagon or disjoint empty convex pentagon and empty convex quadrilateral in the remaining cases, that is, when there is a set of 11 point with at most five points in the convex hull and at least two points in the third convex layer.

Keywords. Discrete geometry, Empty convex polygons, Erdős-Szekeres theorem, Ramsey-type results.

1 Introduction

While addressing the problem of pseudo-convex decomposition, Aichholzer et al. [1] proved the following two theorems using the order type data base [2, 3].

Theorem 1 *Every set of 8 points in the plane, no three on a line, either contains a 5-hole or two disjoint 4-holes.*

Theorem 2 *Every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole.*

Their paper contains a simple geometric proof of Theorem 1 based on a few case distinctions. However, Theorem 2 is only verified using the order type data base [2, 3]. In this two part paper, we give an intuitive geometric proof of Theorem 2 which requires the careful analysis of a moderate number of case distinctions. We begin the first part with some basic observations. Using them we show that any set of 11 points in the plane, no three on a line, with at most six points in the convex hull and exactly one point in the third convex layer contains an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. Finally, we show that any set of 11 points in the plane with at least six points in the convex hull also contains an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. In this part we show the existence of an empty convex hexagon or disjoint empty convex pentagon and empty convex quadrilateral in the remaining cases, that is, when there are at most five points in the convex hull and at least two points in the third convex layer.

2 Notations and Definitions

We first introduce the definitions and notations required for the remainder of the paper. Let S be a finite set of points in the plane in general position, that is, no three on a line. Denote the convex hull of S by $CH(S)$. The boundary vertices of $CH(S)$, and the points of S in the interior of $CH(S)$ are denoted by $\mathcal{V}(CH(S))$ and $\mathcal{I}(CH(S))$, respectively. A region R in the plane is said to be empty in S if R contains no elements of S in the interior. Moreover, for any set T, $|T|$ denotes the cardinality of T.

By $P := p_1p_2 \ldots p_k$ we denote a convex k-gon with vertices $\{p_1, p_2, \ldots, p_k\}$ taken in anti-clockwise order. $\mathcal{V}(P)$ denotes the set of vertices of P and $\mathcal{I}(P)$ the interior of P. The collection of all points $q \in \mathbb{R}^2$ such that $\{q\} \cup \mathcal{V}(P)$ form a convex $(k+1)$-gon is called the forbidden zone of P.

The *j-th convex layer* of S, denoted by $L\{j, S\}$, is the set of points that lie on the boundary of $CH(S \backslash \{\bigcup_{i=1}^{j-1} L\{i, S\}\})$, where $L\{1, S\} = \mathcal{V}(CH(S))$. $|L\{j, S\}|$ denotes the number of points of S in j-th convex layer and $\mathcal{I}(L\{j, S\})$ is the set of points in S which lies in the interior of $CH(L\{j, S\})$. If $p,\ q \in S$ be such that pq is an edge of the convex hull of the j-th layer, then the open halfplane bounded by the line pq and not containing any point of $S \backslash \{\bigcup_{i=1}^{j-1} L\{i, S\}\}$ will be referred to as the *outer* halfplane induced by the edge pq. For any three points $p, q, r \in S$, $\mathcal{H}(pq, r)$ (respectively $\mathcal{H}_c(pq, r)$) denotes the open (respectively closed) halfplane bounded by the line pq containing the point r. Similarly, $\overline{\mathcal{H}}(pq, r)$ (respectively $\overline{\mathcal{H}}_c(pq, r)$) is the open (respectively closed) halfplane bounded by pq not containing the point r. Moreover, if $\angle rpq < \pi$, $Cone(rpq)$ denotes the interior of the angular domain $\angle rpq$.

3 Proof of Theorem 2 Continued

Recall that a set of 11 points is *admissible* if it contains either a 6-hole or a 5-hole and a 4-hole which are disjoint. Let S be a set 11 points in general position. Let $|L\{1, S\}| = |CH(S)| = k$ and $|L\{2, S\}| = |CH(\mathcal{I}(CH(S)))| = m$ with $\mathcal{V}(CH(S)) = L\{1, S\} = \{s_1, s_2, \ldots, s_k\}$ and $L\{2, S\} = \{p_1, p_2, \ldots, p_m\}$, where the vertices are taken in counter-clockwise order. While indexing a set of points from $L\{1, S\}$ or $L\{2, S\}$, we identify indices modulo k or modulo m, respectively.

The following results were proved in the first part of the paper. We shall use them for proving the remaining cases.

Lemma 1 *[4] Any set S of 11 points in general position, with $|CH(S)| \leq 6$ and $|L\{3, S\}| = 1$, is admissible.*

Lemma 2 *[4] S is admissible whenever $|CH(S)| \geq 6$.*

We continue the proof of Theorem 2 with the following lemma.

Lemma 3 *S is admissible whenever $|L\{2, S\}| \geq 6$.*

Proof. Since $|CH(S)| \geq 3$ and $|S| = 11$, $L\{3, S\} \leq 2$. S contains a 6-hole whenever $L\{3, S\} = 0$. Moreover, admissibility of S follows from Lemma 1, whenever $|L\{3, S\}| = 1$. Therefore, it suffices to assume that $|L\{3, S\}| = 2$, with $p, q \in L\{3, S\}$. Now, S contains a 6-hole if $|\mathcal{H}(pq, p_1) \cap L\{2, S\}| \geq 4$ or $|\overline{\mathcal{H}}(pq, p_1) \cap L\{2, S\}| \geq 4$. Therefore, assume $|\mathcal{H}(pq, p_1) \cap L\{2, S\}| = |\overline{\mathcal{H}}(pq, p_1) \cap L\{2, S\}| = 3$. Since $|CH(S)| = 3$, either $|\mathcal{H}(pq, p_1) \cap S| = 5$ or $|\overline{\mathcal{H}}(pq, p_1) \cap S| = 5$. Without loss of generality, let $|\mathcal{H}(pq, p_1) \cap S| = 5$. Then $\overline{\mathcal{H}}_c(pq, p_1) \cap S$ forms a 5-hole which is disjoint from the 4-hole contained in $\mathcal{H}(pq, p_1) \cap S$, since $H(4) = 5$. □

Now, it remains to prove the admissibility of S only when $3 \leq |L\{2, S\}| \leq 5$. The three different cases which arise are presented separately in the following lemmas.

Lemma 4 *S is admissible whenever $|L\{2, S\}| = 5$.*

Proof. Lemma 2 implies that we only need to consider the cases where $3 \leq |CH(S)| \leq 5$, that is, $1 \leq |L\{3,S\}| \leq 3$. Moreover, if $|L\{3,S\}| = 1$, the admissibility of S follows from Lemma 1. So, we have the following two cases:

$|L\{3,S\}| = 2$: Let $p, q \in \mathcal{V}(L\{3,S\})$. If either $|\mathcal{H}(pq,p_1) \cap L\{2,S\}| = 4$ or $|\overline{\mathcal{H}}(pq,p_1) \cap L\{2,S\}| = 4$, S contains a 6-hole. So, let $|\overline{\mathcal{H}}(pq,p_1) \cap S| = 2$ with $\overline{\mathcal{H}}(pq,p_1) = \{p_3,p_4\}$. Consider the partition of the exterior of the pentagon $L\{2,S\}$ into disjoint regions R_i as shown in Figure 1(a). S is admissible unless $|R_4| \leq 1$ and $|R_3| + |R_5| \leq 2$. Moreover, the forbidden zone of the 5-hole $p_1p_2pqp_5$ can be assumed to be empty. Thus, $|R_2| = |R_6| = 0$, which implies that $|R_1| > 0$.

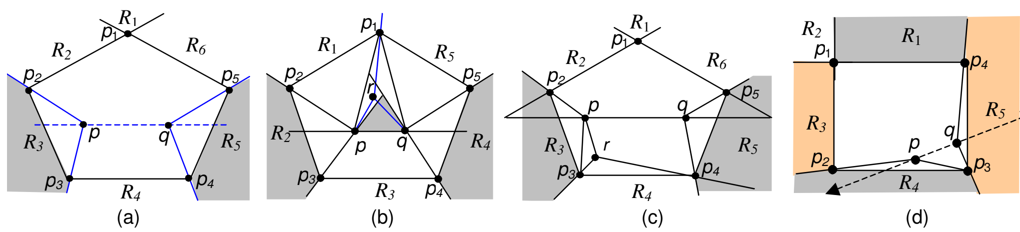

Figure 1: Illustration of the proof of Lemma 4: (a) $|L\{3,S\}| = 2$, (b) $|L\{3,S\}| = 3$ and $|\overline{\mathcal{H}}(pq,r) \cap L\{2,S\}| = 2$, (c) $|L\{3,S\}| = 3$ and $|\overline{\mathcal{H}}(pq,r) \cap L\{2,S\}| = 3$, (d) Illustration of the proof of Lemma 5 when $|\mathcal{I}(L\{2,S\})| = 2$ and $|\mathcal{H}(pq,p_1) \cap L\{2,S\}| = 3$.

$|R_1| = 2$: If there exists a point of S in the forbidden zone of the 4-hole pp_3p_4q, then pp_3p_4q can be extended to a 5-hole which is disjoint from the 4-hole contained in $\overline{\mathcal{H}}_c(p_1p_2,p_3)$. So, assume that the forbidden zone of the 4-hole pp_3p_4q is empty in S. Also, since the forbidden zone of $p_1p_2pqp_5$ is empty in S and $|\overline{\mathcal{H}}(p_1p_2,p_3) \cap S| = |\overline{\mathcal{H}}(p_2p_3,p_1) \cap S| = 2$, we have $|\overline{\mathcal{H}}(p_3p_4,p) \cap S| = 2$. Then, $p_1p_2pqp_5$ is a 5-hole which is disjoint from the 4-hole contained in $\overline{\mathcal{H}}_c(p_3p_4,p) \cap S$.

$|R_1| = 1$: This implies that $|R_4| = 1$ and S contains a 5-hole and a disjoint 4-hole, unless $|\overline{\mathcal{H}}(p_1p_2,p) \cap S| = |\overline{\mathcal{H}}(p_1p_5,p) \cap S| = 1$. Moreover, since the forbidden zone of the 5-hole $p_1p_2pqp_5$ can be assumed to be empty, there must be two points of $\mathcal{V}(CH(S))$ in the $R_3 \cup R_5$ which also lie in the forbidden zone of the 4-hole pp_3p_4q. Then, $((R_3 \cup R_5) \cap S) \cup \{p,p_3,p_4,q\}$ contains a 6-hole.

$|L\{3,S\}|=3$**:** Let $p,q,r\in\mathcal{V}(L\{3,S\})$. Now, we consider the following two cases:

$|\overline{\mathcal{H}}(pq,r)\cap L\{2,S\}|=2$**:** Let $\overline{\mathcal{H}}(pq,r)\cap L\{2,S\}=\{p_3,p_4\}$ (see Figure 1(c)). If $r\notin\mathcal{I}(\triangle p_1pq)$, the admissibility of S can be verified easily. Therefore, assume that $r\in\mathcal{I}(\triangle p_1pq)$ and consider the partition of the exterior of $L\{2,S\}$ into disjoint regions R_i as shown in Figure 1(b). S is admissible unless $|R_1|+|R_4|\leq 1$ and $|R_2|+|R_5|\leq 1$. This implies that $|R_3|=1$, and therefore $|R_1|=|R_5|=0$. Moreover, S is admissible unless $|R_1|+|R_4|\leq 2$, which implies that $|R_2|=|R_4|=1$. Now, if $r\in\mathcal{H}(pq,p_1)\cap\overline{\mathcal{H}}(p_3p,p_1)\cap\overline{\mathcal{H}}(p_4q,p_1)$, we have S clearly contains a 6-hole. Otherwise, either $p_3prp_1p_2$ is a 5-hole which is disjoint from the 4-hole contained in $\{p_4,p_5,q\}\cup(R_4\cap S)$ or $p_4qrp_1p_5$ is a 5-hole disjoint from the 4-hole contained in $\{p_2,p_3,p\}\cup(R_2\cap S)$.

$|\overline{\mathcal{H}}(pq,r)\cap L\{2,S\}|=3$**:** Let $\overline{\mathcal{H}}(pq,r)\cap L\{2,S\}=\{p_1,p_2,p_5\}$. The admissibility of S can be verified easily if $r\notin\mathcal{I}(p_3pqp_4)$. So, let $r\in\mathcal{I}(p_3pqp_4)$ and consider the partition the partition of the exterior of $p_1p_2p_3p_4p_5$ into disjoint regions R_i as shown in Figure 1(c). Now, S is admissible unless $|R_3|+|R_5|\leq 1$ and $|R_4|=0$, which implies $|R_1|=2$. If $r\in\mathcal{H}(p_2p,p_3)$, then $|R_3|=0$ and $\{p,q,r,p_4\}\cup(R_5\cap S)$ forms a 5-hole which is disjoint from the 4-hole formed by $(R_1\cap S)\cup\{p_1,p_5\}$. Otherwise, $r\in\mathcal{H}(p_5q,p_4)$ and the admissibility of S follows from similar arguments. □

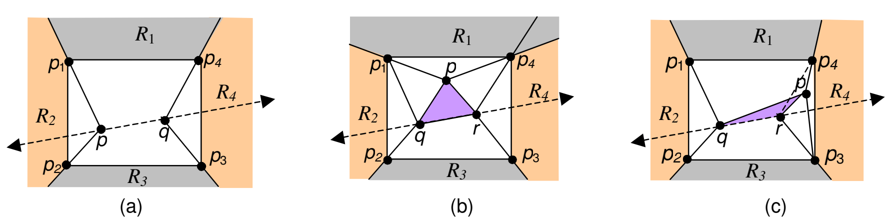

Figure 2: Illustration of the proof of Lemma 5

Lemma 5 *S is admissible whenever $|L\{2,S\}|=4$ and $|\mathcal{I}(L\{2,S\})|\leq 3$.*

Proof. Since $|S|=11$ and $|CH(S)|\leq 5$ from Lemma 2, we have $1\leq|\mathcal{I}(L\{2,S\})|\leq 4$. Lemma 1 implies that S is admissible whenever $|\mathcal{I}(L\{2,S\})|=1$. We consider the remaining two cases separately.

$|\mathcal{I}(L\{2,S\})|=2$**:** Let $p,q\in\mathcal{I}(L\{2,S\})$. We have two cases:

$|\mathcal{H}(pq, p_1) \cap L\{2, S\}| = 3$**:** Assume that $\mathcal{H}(pq, p_1) \cap L\{2, S\} = \{p_1, p_2, p_4\}$ and $\overline{\mathcal{H}}(pq, p_1) \cap L\{2, S\} = \{p_3\}$. Consider the partition of the exterior of $L\{2, S\}$ into disjoint regions R_i as shown in Figure 1(d). Clearly $|R_1| = |R_3| = 0$, otherwise S contains a 6-hole. Moreover, S is admissible whenever $|R_4| + |R_5| \geq 4$. Therefore, $|R_4| + |R_5| \leq 3$, which implies that $|R_2| = 2$ by Observation 3 of Part I [4]. Now, if $|R_4| = 2$, then $(R_4 \cap S) \cup \{p, p_2, p_3\}$ forms a 5-hole which is disjoint from the 4-hole in $(R_2 \cap S) \cup \{p_1, p_4\}$. Therefore, assume $|R_4| \leq 1$, and similarly $|R_5| \leq 1$. This implies, $\sum_{i=1}^{5} |R_i| \leq 4 < 5 = |\mathcal{V}(CH(S))|$, which is a contradiction.

$|\mathcal{H}(pq, p_1) \cap L\{2, S\}| = 2$**:** Assume that $\mathcal{H}(pq, p_1) \cap L\{2, S\} = \{p_1, p_4\}$ and $\overline{\mathcal{H}}(pq, p_1) \cap L\{2, S\} = \{p_2, p_3\}$. Consider the partition of the exterior of $L\{2, S\}$ into disjoint regions R_i as shown in Figure 2(a). Now, S is admissible unless $|R_1| \leq 1$, $|R_3| \leq 1$, and $|R_2| + |R_4| \leq 2$. This implies, $\sum_{i=1}^{4} |R_i| \leq 4$, which contradicts $|\mathcal{V}(CH(S))| = 5$.

$|\mathcal{I}(L\{2, S\})| = 3$**:** Let $p, q, r \in \mathcal{I}(L\{2, S\})$. It is easy to see that S contains a 6-hole when no outer halfpane induced by the edges of $L\{3, S\}$ contains two points of $L\{2, S\}$. So, assume $|\overline{\mathcal{H}}(qr, p) \cap L\{2, S\}| = 2$ with $\overline{\mathcal{H}}(qr, p) \cap L\{2, S\} = \{p_2, p_3\}$. Now, we consider the following cases:

$p \in \mathcal{I}(qp_1p_4r)$**:** Consider the partition of the exterior of $p_1p_2p_3p_4$ into disjoint regions R_i as shown in Figure 2(b). Observe that S is admissible unless $|R_2| + |R_4| \leq 2$ and $|R_1| + |R_3| \leq 1$. This implies, $\sum_{i=1}^{4} |R_i| \leq 3 < 4$, which is impossible.

$p \notin \mathcal{I}(qp_1p_4r)$**:** Consider the partition of the exterior of $p_1p_2p_3p_4$ into regions R_i as shown in Figure 2(c). S contains a 6-hole unless $|R_1| = 0$ and $|R_3| \leq 1$. Moreover, S is admissible if $|R_2| + |R_4| \geq 3$. However, if $|R_2| + |R_4| \leq 2$, then $\sum_{i=1}^{4} |R_i| = 3$, which contradicts $|L\{2, S\}| = 4$. □

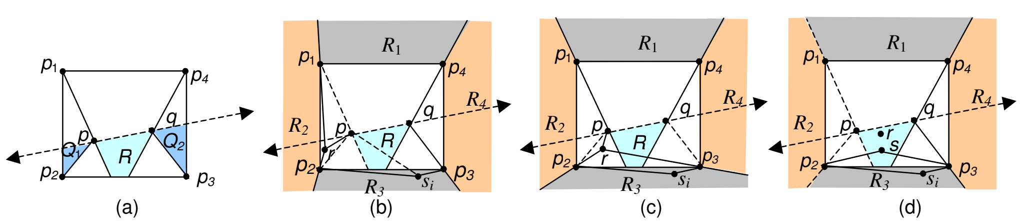

Figure 3: Illustration of the proof of Lemma 6

Lemma 6 *S is admissible whenever $|L\{2,S\}| = 4$ and $|\mathcal{I}(L\{2,S\})| = 4$.*

Proof. Let $p, q, r, s \in \mathcal{I}(L\{2,S\})$ such that pq is an edge of $L\{3,S\}$. It can be shown that S is admissible if none of the outer halfplane induced by the edges of the third layer contains two points of $L\{2,S\}$. First, assume that $\overline{\mathcal{H}}(pq,r) \cap L\{2,S\} = \{p_1, p_4\}$. Let the regions Q_1, Q_2, and R, be as shown in Figure 3(a). Note that R is the part the forbidden zone of the 4-hole p_1pqp_4 that lies in the halfplane $\mathcal{H}(pq, p_2)$.

To begin with suppose that $(Q_1 \cup Q_2) \cap S$ non-empty, and let $r \in Q_1$. If $\{p, q, r, s, p_2, p_3\}$ forms a 6-hole, then S is admissible. Otherwise, we can assume that $pqrp_2p_3$ is a convex pentagon with s in its interior. Consider the partition of the exterior of $p_1p_2p_3p_4$ as shown in Figure 3(b). If $|R_1| \geq 2$, then $(R_1 \cap S) \cup \{p, q, p_1, p_4\}$ contains a 6-hole. Also, if $|R_1| = 1$, the 5-hole formed by $(R_1 \cap S) \cup \{p, q, p_1, p_4\}$ is disjoint from the 4-hole contained in $\mathcal{H}(pq, p_2) \cap S$, since $|\mathcal{H}(pq, p_2) \cap S| \geq 5$. Next, consider $|R_2| = 2$, then $(R_2 \cap S) \cup \{p_1, p_2, r\}$ contains a 5-hole. This 5-hole is disjoint from the 4-hole contained in $(R_4 \cap S) \cup \{q, p_3, p_4\}$ when $R_4 \cap S$ is non-empty, or the 4-hole contained in $(R_3 \cap S) \cup \{p, q, p_3\}$ when $R_3 \cap S$ is non-empty. Therefore, $|R_2| \leq 1$ and similarly $|R_4| \leq 1$. Finally, if $|R_3| \geq 2$, the 5-hole formed by in $(R_3 \cap S) \cup \{p_2, p_3, s\}$ is disjoint from the 4-hole pqp_1p_4. This implies that $|R_2| = |R_3| = |R_4| = 1$, and let $R_3 \cap S = \{s_i\}$. If $s \in \mathcal{I}(prp_2s_i)$, the 5-hole pqp_3s_is is disjoint from the 4-hole contained in $(R_2 \cap S) \cup \{p_1, p_2, r\}$. Similarly, if $s \in \mathcal{I}(p_3s_ipq)$, prp_2s_is is a 5-hole which is disjoint from the 4-hole contained in $(R_4 \cap S) \cup \{p_3, p_4, q\}$.

Otherwise, $Q_1 \cup Q_2$ is empty in S. Depending on the location of the points r and s, we consider the partition of the exterior of $p_1p_2p_3p_4$ into disjoint regions R_i as shown in Figure 3(c) and Figure 3(d). The admissibility of S now follows from similar arguments as before. □

Before, proceeding to the final part of the proof, we recall the definition of a pseudo-triangle. A pseudo-triangle is a simple polygon having exactly three convex vertices. Therefore, a pseudo-triangle has three convex vertices connected by concave side chains that consists of one or more edges. A pseudo-triangle with s-vertices is called a s-pseudo-triangle. A pseudo-triangle is said to be *standard* if each side chain has at least two edges.

Observation 1 *Let $T = abc$ be a triangle with at least three interior points. If Z denotes the set of all points inside the triangle T, then there exists three points $p, q, r \in Z$ such that $apbqcr$ is standard 6-pseudo-triangle and $\mathcal{I}(\triangle apb) \cap Z$, $\mathcal{I}(\triangle bqc) \cap Z$, and $\mathcal{I}(\triangle cra) \cap Z$ are empty.*

Proof. It follows from the results proved in [1], that there exists three points $p', q', r' \in Z$ such that $ap'bq'cr'$ is a standard 6-pseudo-triangle. If $\mathcal{I}(\triangle ap'b) \cap Z$ is non-empty, there exists $p \in \mathcal{I}(\triangle ap'b) \cap Z$ such that $\mathcal{I}(\triangle apb) \cap Z$ is empty.

Otherwise, $\mathcal{I}(\triangle ap'b) \cap Z$ is empty and we take $p = p'$. The points q and r are chosen similarly from $\mathcal{I}(\triangle aq'b) \cap Z$ and $\mathcal{I}(\triangle ar'b) \cap Z$, respectively. Observe that $apbqcr$ is standard 6-pseudo-triangle and $\mathcal{I}(\triangle apb) \cap Z$, $\mathcal{I}(\triangle bqc) \cap Z$, and $\mathcal{I}(\triangle cra) \cap Z$ are empty. □

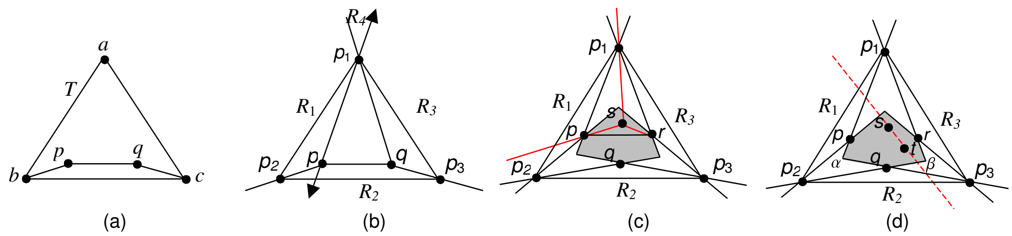

Figure 4: (a) Points p and q are comparable, Illustration of the proof of Lemma 7: (b) $|\mathcal{I}(L\{2, S\})| = 4$ and the points p and q are comparable, (c) $|\mathcal{I}(L\{2, S\})| = 4$ and every pair points in $\mathcal{I}(L\{2, S\})$ are incomparable, (d) $|\mathcal{I}(L\{2, S\})| = 5$.

Lemma 7 *S is admissible whenever $|L\{2, S\}| = 3$.*

Proof. Now, we consider the different cases based on the number of points of S inside $L\{2, S\}$. We say two points $p, q \in \mathcal{I}(L\{2, S\})$ are *comparable* if there exits an edge $e = p_i p_{i+1}$ of $L\{2, S\}$ such that $\{p_i, p_{i+1}, p, q\}$ forms a 4-hole. Two points $p, q \in \mathcal{I}(L\{2, S\})$ are said to be *incomparable* if they are not comparable. For example, in Figure 4(a), the points p and q are comparable with respect to the edge bc of $\triangle abc$, since $pqcb$ is a 4-hole.

$|\mathcal{I}(L\{2, S\})| = 3$**:** Observation 1 ensures that there exists three points $p, q, r \in \mathcal{I}(L\{2, S\})$ such that $p_1 p p_2 q p_3 r$ forms a standard 6-pseudo-triangle. Now, S contains a 6-hole unless $|Cone(p_1 p p_2) \cap S|$, $|Cone(p_2 q p_3) \cap S|$, and $|Cone(p_3 r p_1) \cap S|$ are all less than three. Therefore, without loss of generality, we can assume that $|Cone(p_1 r p_3) \cap S| = 1$, since $|CH(S)| = 5$. Then $(Cone(p_1 p p_2) \cap S) \cup \{p_1, p, p_2\}$ forms a 5-hole which is disjoint from the 4-hole formed by $(Cone(p_2 q p_3) \cap S) \cup \{q, p_3\}$.

$|\mathcal{I}(L\{2, S\})| = 4$**:** Let $\mathcal{I}(L\{2, S\}) = \{p, q, r, s\}$. Based on the comparability of the points in $L\{2, S\}$, we have the following cases:

Case **1:** Two points of $\mathcal{I}(L\{2, S\})$ are comparable. Without loss of generality, assume that p and q are comparable with respect to the edge $e = p_2 p_3$. Consider the partition of the exterior of $L\{2, S\}$ into different disjoint regions R_i as shown in Figure 4(b). Clearly, S contains a 6-hole whenever $|R_2| \geq 2$. Therefore, $|R_2| \leq 1$, and

$|R_1| + |R_3| + |R_4| \geq 3$, since $|CH(S)| = 4$. Now, by the pigeonhole principle either $|R_1| + |R_4| \geq 2$ or $|R_3| + |R_4| \geq 2$. Without loss of generality, let $|R_1| + |R_4| = 2$.

$|R_2| = 0$: This implies, $|R_3| = 2$ and the 5-hole contained in $\{p_1, p_3, q\} \cup (R_3 \cap S)$ is disjoint form the 4-hole contained in $\{p_1, p\} \cup (Cone(p_1pp_2) \cap S)$.

$|R_2| = 1$: If $r \in \mathcal{I}(pp_1p_2) \cap S$, $\{p, p_2, p_3, q\} \cup (R_2 \cap S)$ forms a 5-hole and $\{p_1, r\} \cup ((R_1 \cup R_4) \cap S)$ is a disjoint 4-hole. Otherwise, $\mathcal{I}(pp_1p_2) \cap S$ is empty and $\{p_1, p_2, p\} \cup ((R_1 \cup R_4) \cap S)$ forms a 5-hole disjoint from the 4-hole contained in $\mathcal{H}(p_1p, p_3) \cap S$, since $|\mathcal{H}(p_1p, p_3) \cap S| \geq 5$.

Case **2:** Every pair points in $\mathcal{I}(L\{2, S\})$ are incomparable. Observation 1 ensures that we have a standard 6-pseudo-triangle in $L\{2, S\}$, say $p_1pp_2qp_3r$, such that $\mathcal{I}(p_1pp_2)$, $\mathcal{I}(p_2qp_3)$, and $\mathcal{I}(p_3rp_1)$ are empty in S. Moreover, since every pair of points are incomparable the forbidden zones of p_1pp_2, p_2qp_3, and p_3rp_1 are empty in $\mathcal{I}\{L(2, S\}) \cap S$. Thus, the remaining point $s \in \mathcal{I}(L\{2, S\})$ lies in the shaded region R as indicated in Figure 4(c). As before we may assume that $|Cone(p_1pp_2) \cap S| = |Cone(p_1rp_3) \cap S| = 1$ and $|Cone(p_2qp_3) \cap S| = 2$. If $s \in \overline{\mathcal{H}}(pr, p_1)$, then $\{p_2, p_3, q\} \cup (Cone(p_2qp_3) \cap S)$ contains a 5-hole which is disjoint from the 4-hole p_1psr. Otherwise, $s \in \mathcal{H}(pr, p_1)$, and the admissibility of S is immediate whenever $Cone(p_1sp) \cap S$ is non-empty. Otherwise, $Cone(p_1sp) \cap S$ is empty, and $(R_1 \cap S) \cup \{p_2, p, q, s\}$ contains an 5-hole which is disjoint from the 4-hole contained in $(Cone(p_1rp_3) \cap S) \cup \{p_1, r, p_3\}$.

$|\mathcal{I}(L\{2, S\})| = 5$: Let $\mathcal{I}(L\{2, S\}) = \{p, q, r, s, t\}$. If any two points in $\mathcal{I}(L\{2, S\})$ are comparable, the admissibility of S follows from arguments similar to the analogous case for $|\mathcal{I}(L\{2, S\})| = 4$. Therefore, it suffices to prove the admissibility of S when every pair points in $\mathcal{I}(L\{2, S\})$ are incomparable. Observation 1 ensures that we have a standard 6-pseudo-triangle in $L\{2, S\}$, say $p_1pp_2qp_3r$, such that $\mathcal{I}(p_1pp_2)$, $\mathcal{I}(p_2qp_3)$, and $\mathcal{I}(p_3rp_1)$ are empty in S. Moreover, since every pair of points are incomparable the forbidden zones of p_1pp_2, p_2qp_3, and p_3rp_1 are empty in $\mathcal{I}\{L(2, S\})$. Thus, the remaining points $s, t \in \mathcal{I}(L\{2, S\})$ lies in the shaded region R as indicated in Figure 4(d). Let α, β be as shown in Figure 4(d). Now, if $|Cone(p_2qp_3) \cap S| \geq 2$, the 5-hole contained in $Cone(p_2qp_3) \cap S \cup \{p_2, q, p_3\}$ is disjoint from the 4-hole contained in $Cone(\alpha q\beta) \cap S$, since $|Cone(\alpha q\beta) \cap S| \geq 5$. Therefore, assume that $|Cone(p_1pp_2) \cap S| = |Cone(p_2qp_3) \cap S| = |Cone(p_3rp_1) \cap S| = 1$. Observe that S contains a 6-hole whenever $|\mathcal{H}(st, p) \cap \mathcal{I}(L\{2, S\})| = 3$. Otherwise, let $|\mathcal{H}(st, p) \cap \mathcal{I}(L\{2, S\})| = 2$ with $\mathcal{H}(st, p) \cap \mathcal{I}(L\{2, S\}) = \{p, q\}$

(see Figure 4(d)). Then, p_2pstq is a 5-hole which is disjoint from the 4-hole contained in $(Cone(p_1rp_3) \cap S) \cup \{p_1, p_3, r\}$. □

Since all the different cases have been considered, the proof finally completes.

4 Conclusions

This is the second part of a two part paper where we gave a geometric proof of a Ramsey-type for disjoint empty convex polygons, which states that every set of 11 points in the plane, no three on a line, contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral. This result was first proved by Aichholzer et al. [1] using the computer-aided order type enumeration method. Aichholzer et al. [1] used this theorem to establish bounds on the pseudo-convex decomposition number of planar point sets.

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