On the Minimum Size of a Point Set Containing a 5-Hole and a Disjoint 4-Hole

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Abstract. Let $H(k,l)$ denote the smallest integer such that any set of $H(k,l)$ points in the plane, no three on a line, contains an empty convex k-gon and an empty convex l-gon, which are disjoint, that is, their convex hulls do not intersect. Hosono and Urabe [*JCDCG*, LNCS 3742, 117-122, 2004] proved that $12 \leq H(4,5) \leq 14$. Very recently, using a Ramsey-type result for disjoint empty convex polygons proved by Aichholzer et al. [*Graphs and Combinatorics*, Vol. 23, 481-507, 2007], Hosono and Urabe [*KyotoCGGT*, LNCS 4535, 90-100, 2008] improve the upper bound to 13. In this paper, with the help of the same Ramsey-type result, we prove that $H(4,5) = 12$.

Keywords. Convex hull, Discrete geometry, Empty convex polygons, Erdős-Szekeres theorem, Ramsey-type results.

1 Introduction

The famous Erdős-Szekeres theorem [7] states that for every positive integer m, there exists a smallest integer $ES(m)$, such that any set of at least $ES(m)$ points in the plane, no three on a line, contains m points which lie on the vertices of a convex polygon. Evaluating the exact value of $ES(m)$ is a long standing open problem. A construction due to Erdős [8] shows that $ES(m) \geq 2^{m-2} + 1$, which is also conjectured to be sharp. It is known that $ES(4) = 5$ and $ES(5) = 9$ [15]. Following a long computer search, Szekeres and Peters [19] recently proved that $ES(6) = 17$. The value of $ES(m)$ is unknown for all $m > 6$. The best known upper bound for $m \geq 7$ is due to Toth and Valtr [20]: $ES(m) \leq \binom{2m-5}{m-3} + 1$. For a more detailed description of the Erdős-Szekeres theorem and its numerous ramifications, see the surveys by Bárány and Károlyi [4] and Morris and Soltan [16].

In 1978 Erdős [6] asked whether for every positive integer k, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains k points which lie on the vertices of convex polygon whose interior contains no points of the set. Such a subset is called an *empty convex k-gon* or a *k-hole*. Esther Klein showed $H(4) = 5$ and Harborth [10] proved that $H(5) = 10$. Horton [11] showed that it is possible to construct arbitrarily large set of points without a 7-hole, thereby proving that $H(k)$ does not exist for $k \geq 7$. Recently, after a long wait, the existence of $H(6)$ has been proved by Gerken [9] and independently by Nicolás [17]. Later Valtr [22] gave a simpler version of Gerken's proof.

The problems concerning disjoint holes, that is, empty convex polygons with disjoint convex hulls, was first studied by Urabe [21] while addressing the problem of partitioning of planar point sets. For any set S of points in the plane, denote by $CH(S)$ the *convex hull* of S. Given a set S of n points in the plane, no three on a line, a *disjoint convex partition* of S is a partition of S into subsets $S_1, S_2, \ldots S_t$, with $\sum_{i=1}^{t} |S_i| = n$, such that for each $i \in \{1, 2, \ldots, t\}$, $CH(S_i)$ forms a $|S_i|$-gon and $CH(S_i) \cap CH(S_j) = \emptyset$, for any pair of indices

i, j. Observe that in any disjoint convex partition of S, the set S_i forms a $|S_i|$-hole and the holes formed by the sets S_i and S_j are disjoint for any pair of distinct indices i, j. If $F(S)$ denote the minimum number of disjoint holes in any disjoint convex partition of S, then $F(n) = \max_S F(S)$, where the maximum is taken over all sets S of n points, is called the *disjoint convex partition number* for all sets of fixed size n. The disjoint convex partition number $F(n)$ is bounded by $\lceil \frac{n-1}{4} \rceil \leq F(n) \leq \lceil \frac{5n}{18} \rceil$. The lower bound is by Urabe [21] and the upper bound by Hosono and Urabe [14]. The proof of the upper bound uses the fact that every set of 7 points in the plane contains a 3-hole and a disjoint 4-hole. Later, Xu and Ding [25] improved the lower bound to $\lceil \frac{n+1}{4} \rceil$.

Another class of related problems arise if the condition of disjointness is relaxed. Given a set S of n points in the plane, no three on a line, a *empty convex partition* of S is a partition of S into subsets $S_1, S_2, \ldots S_t$, with $\sum_{i=1}^{t} |S_i| = n$, such that for each $i \in \{1, 2, \ldots, t\}$, $CH(S_i)$ forms a $|S_i|$-hole in S. In this case, $CH(S_i)$ and $CH(S_j)$ may intersect for some pair of distinct indices i and j. If $G(S)$ denote the minimum number of holes in any empty convex partition of S, then the *empty convex partition number* for all sets of fixed size n is $G(n) = \max_S G(S)$, where the maximum is taken over all sets S of n points. Urabe [21] proved that $\lceil \frac{n-1}{4} \rceil \leq G(n) \leq \lceil \frac{3n}{11} \rceil$. Xu and Ding [25] improved the bounds to $\lceil \frac{n+1}{4} \rceil \leq G(n) \leq \lceil \frac{5n}{14} \rceil$. The upper bound bound was further improved to $\lceil \frac{9n}{34} \rceil$ by Ding et al. [5].

In [14], Urabe defined the function $F_k(n) = \min_S F_k(S)$, where $F_k(S)$ is the maximum number of k-holes in a disjoint convex partition of S, and the the minimum being taken over all sets S of n points. Using the fact that the minimum size of a point set containing two disjoint 4-holes is 9, they showed that $F_4(n) \geq \lfloor \frac{5n}{22} \rfloor$. Recently, Wu and Ding [23] defined $G_k(n) = \min_S G_k(S)$, where $G_k(S)$ is the maximum number of k-holes in a empty convex partition of S, and the the minimum being taken over all sets S of n points. They proved that $G_4(n) \geq \lfloor \frac{9n}{38} \rfloor$. The problem of obtaining non-trivial lower bounds on $F_5(n)$ and $G_5(n)$ remains open.

Hosono and Urabe [13] also introduced the function $H(k, l)$, $k \leq l$, which denotes the smallest integer such that any set of $H(k, l)$ points in the plane, no three on a line, contains both a k-hole and a l-hole which are disjoint. Clearly, $H(3, 3) = 6$ and Horton's result [11] implies that $H(k, l)$ does not exist for all $l \geq 7$. Urabe [21] showed that $H(3, 4) = 7$, while Hosono and Urabe [14] showed that $H(4, 4) = 9$. Hosono and Urabe [13] also proved that $H(3, 5) = 10$ and $12 \leq H(4, 5) \leq 14$. The results $H(3, 4) = 7$ and $H(4, 5) \leq 14$ were later reconfirmed by Wu and Ding [24]. Very recently, using a Ramsey-type result for disjoint empty convex polygons proved by Aichholzer et al. [1], Hosono and Urabe [12] proved that $12 \leq H(4, 5) \leq 13$, thus improving upon their earlier result.

In this paper, using the same Ramsey-type result, we evaluate the exact value of $H(4, 5)$, thereby improving upon the result of Hosono and Urabe [12], as stated in the following theorem.

Theorem 1. $H(4, 5) = 12$.

While addressing the problem of pseudo-convex decomposition, Aichholzer et al. [1] proves the following theorem with the help of the order type data base ([2], [3]). Here, we use this result to prove Theorem 1.

Theorem 2. *[1] Every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole.*

The outline of the proof of Theorem 1 is as follows. Consider a set S of 12 points in the plane, no three on a line. Theorem 2 implies that S always contains a 6-hole or a 5-

hole and a 4-hole, which are disjoint. If S contains a 5-hole and a disjoint 4-hole, we are done. Therefore, it suffices to assume that S contains a 6-hole. Next, we show that if S contains a 7-hole, then S contains a 5-hole and a disjoint 4-hole. Thus, we assume that S contains a 6-hole, which cannot be extended to a 7-hole. Then we consider a subdivision of the exterior of the 6-hole and prove the existence a 5-hole and a disjoint 4-hole for all the different possible distributions of the remaining 6 points in the regions formed by the subdivision. The formal proof of Theorem 1 is presented in Section 3.

2 Definitions and Notations

We first introduce the definitions and notations required for the remaining part of the paper. Let S be a finite set of points in the plane in general position, that is, no three on a line. Denote the *convex hull* of S by $CH(S)$. The boundary vertices of $CH(S)$, and the points of S in the interior of $CH(S)$ are denoted by $\mathcal{V}(CH(S))$ and $\mathcal{I}(CH(S))$, respectively. A region R in the plane is said to be *empty* in S if R contains no elements of S in its interior. Moreover, for any set T, $|T|$ denotes the cardinality of T.

By $P := p_1p_2 \ldots p_k$ we denote a convex k-gon with vertices $\{p_1, p_2, \ldots, p_k\}$ taken in anti-clockwise order. $\mathcal{V}(P)$ denotes the set of vertices of P and $\mathcal{I}(P)$ the interior of P. The collection of all points $q \in \mathbb{R}^2$ such that $\{q\} \cup \mathcal{V}(P)$ form a convex $(k+1)$-gon is called the *forbidden zone* of P. The forbidden zone of the pentagon $P := p_1p_2p_3p_4p_5$ is the shaded region as shown in Figure 1(a).

For any three points $p, q, r \in S$, $\mathcal{H}(pq, r)$ denotes the open halfplane bounded by the line pq containing the point r. Similarly, $\overline{\mathcal{H}}(pq, r)$ is the open halfplane bounded by pq not containing the point r. Moreover, if $\angle rpq < \pi$, $Cone(rpq)$ denotes the interior of the angular domain $\angle rpq$. A point $s \in Cone(rpq) \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in $Cone(rpq)$ if $Cone(spq)$ is empty in S. Similarly, for any convex region R a point $s \in R \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in R if $Cone(spq) \cap R$ is empty in S. More generally, for any positive integer k, a point $s \in S$ is called the *k-th angular neighbor* of $\overrightarrow{pq}$ whenever $Cone(spq) \cap R$ contains exactly $k-1$ points of S in its interior.

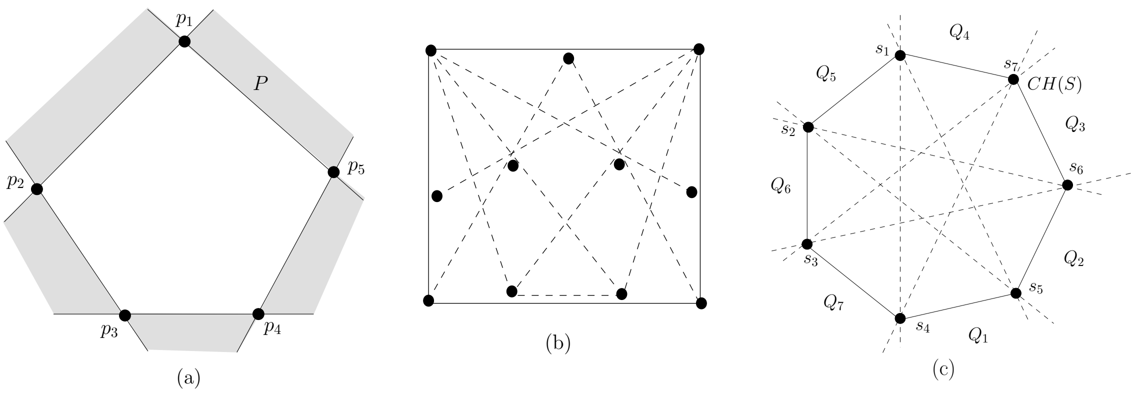


Fig. 1. (a) Forbidden zone of a pentagon P, (b) 11 points without a 5-hole and a 4-hole which are disjoint [13], and (c) Illustration of the proof of Observation 1.

3 Proof of Theorem 1

Urabe and Hosono [13] constructed a set of 11 points not containing an 4-hole and a disjoint 5-hole, which is shown in Figure 1(b). This implies that $H(4,5) \geq 12$. Therefore, for proving the theorem it suffices to show that $H(4,5) \leq 12$.

Let S be a set of 12 points in general position in the plane. We say S is *admissible* whenever S contains a 4-hole and 5-hole which are disjoint.

First, consider that S does not contain a 6-hole. Then Theorem 2 implies that S must contain a 5-hole and a disjoint 4-hole. Therefore, assume that S contains a 6-hole.

We now have the following observation:

Observation 1 *If S contains a 7-hole, then S is admissible.*

Proof. Let $H := s_1s_2s_3s_4s_5s_6s_7$ be a 7-hole in S. For $i \in \{1, 2, \ldots, 7\}$, let Q_i denote the region $Cone(s_{i+3}s_is_{i+4}) \backslash \mathcal{I}(s_{i+3}s_is_{i+4})$ (Figure 1(c)), with indices taken modulo 7. If $|Q_1 \cap S| = 0$, then by the pigeon-hole principle either $|\mathcal{H}(s_1s_4, s_2) \cap S| \geq 5$ or $|\overline{\mathcal{H}}(s_1s_5, s_2) \cap S| \geq 5$. Without loss of generality, let $|\mathcal{H}(s_1s_4, s_2) \cap S| \geq 5$. Then $\mathcal{H}(s_1s_4, s_2) \cap S$ contains a 4-hole, since $H(4) = 5$. This 4-hole is disjoint from the 5-hole $s_1s_4s_5s_6s_7$. Therefore, whenever $|Q_i \cap S| = 0$ for some $i \in \{1, 2, \ldots, 7\}$, then S is admissible. However, $|Q_i \cap S| \geq 1$ for all $i \in \{1, 2, \ldots, 7\}$ implies, $\sum_{i=1}^{7} |Q_i \cap S| \geq 7 > 5 = |S| - |\mathcal{V}(H)|$, which is a contradiction. □

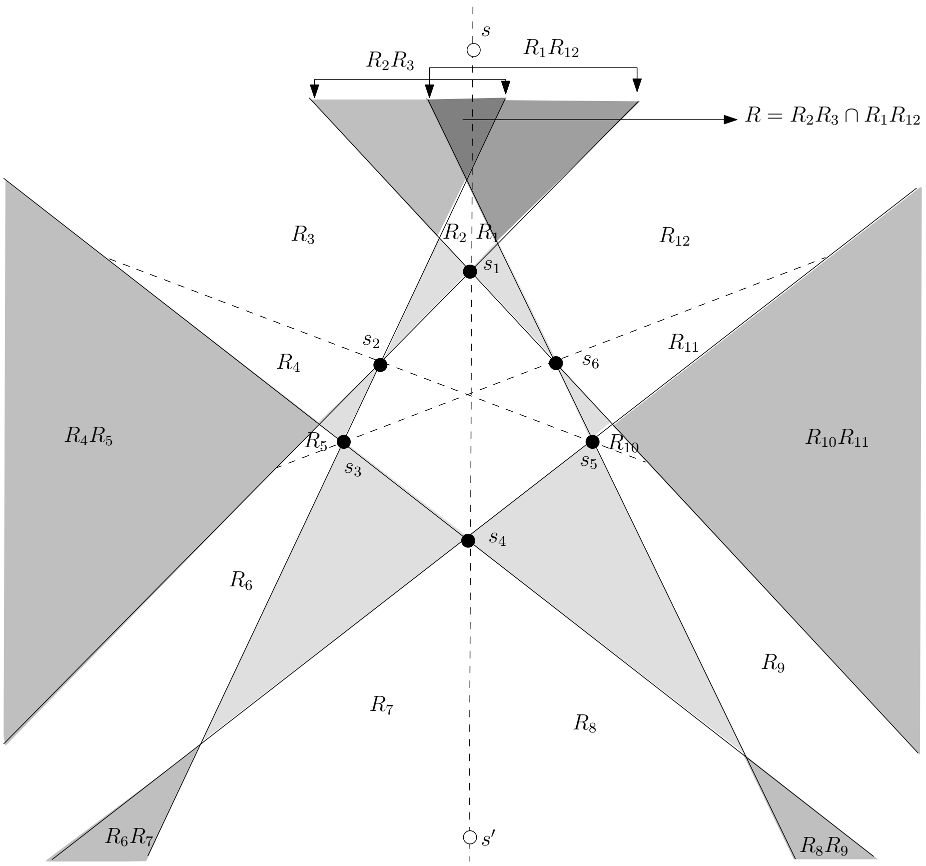


Fig. 2. The subdivision of the exterior of the 6-hole $s_1s_2s_3s_4s_5s_6$.

Let $B := s_1s_2s_3s_4s_5s_6$ be a 6-hole in S. In light of Observation 1, it can be assumed that the forbidden zone of B is empty in S, that is, B cannot be extended to a 7-hole. Hereafter, while indexing the points of $\mathcal{V}(B)$, we identify the indices modulo 6.

We begin with a simple observation:

Observation 2 *If for some $s_i \in \mathcal{V}(B)$, $|\overline{\mathcal{H}}(s_is_{i+1}, s_{i+2}) \cap S| \geq 4$, then S is admissible.*

Proof. If $|\overline{\mathcal{H}}(s_is_{i+1}, s_{i+2}) \cap S| \geq 4$, then $(\overline{\mathcal{H}}(s_is_{i+1}, s_{i+2}) \cap S) \cup \{s_i\}$ contains a 4-hole, as $H(4) = 5$. This 4-hole is disjoint from the 5-hole formed by $\mathcal{V}(B) \backslash \{s_i\}$. Hence S is admissible. □

Consider the subdivision of the exterior of the hexagon B into regions R_i and R_iR_j, as shown in Figure 2. The regions of the type R_i are disjoint from each other, but the regions of the type R_iR_j may overlap with each other but are disjoint from regions of the type R_i. Observe that in Figure 2, the deeply shaded region R is the intersection of the regions R_2R_3 and R_1R_{12}. $|R_i|$ or $|R_iR_j|$ denotes the number of points of S in R_i or R_iR_j, respectively. Also, let s, s' be two points on the extended line s_1s_4 as shown in Figure 2.

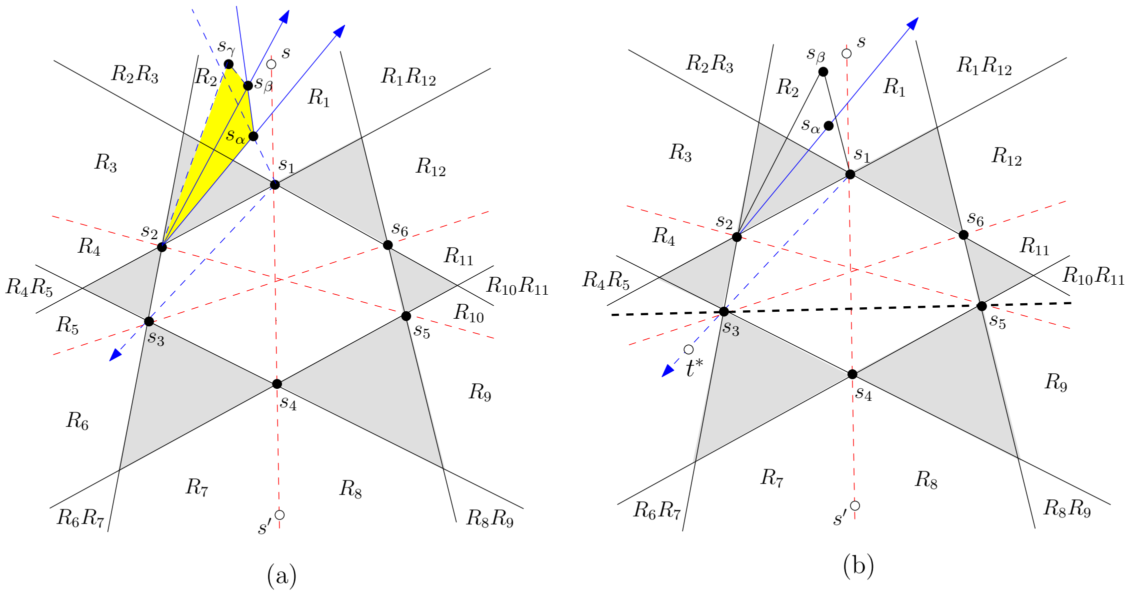


Fig. 3. Illustration of the proof of (a) Observation 3 and (b) Observation 4.

Note, Observation 2 implies that for all $s_i \in \mathcal{V}(B)$, $|\overline{\mathcal{H}}(s_is_{i+1}, s_{i+2}) \cap S| \leq 3$. In particular, $|R_i| \leq 3$ for all $i \in \{1, 2, \ldots, 12\}$. Now, we have the following observations:

Observation 3 *S is admissible, if $|R_i| = 3$ for some $i \in \{1, 2, \ldots, 12\}$.*

Proof. Without loss of generality assume $|R_2| = 3$. Let $s_\alpha \in S$ be the nearest angular neighbor of $\overrightarrow{s_2s_1}$ in R_2. Observation 2 implies that S is admissible unless $|(\mathcal{H}(s_1s_2, s_\alpha) \backslash R_2) \cap S| = |(\mathcal{H}(s_1s_6, s_\alpha) \backslash R_2) \cap S| = 0$.

Case **1:** *The forbidden zone of the 5-hole $s_\alpha s_2s_3s_4s_1$ is empty in $R_2 \cap S$.* Let s_β and s_γ be the other two points in $R_2 \cap S$ such that s_β is the nearest angular neighbor of $\overrightarrow{s_2s_\alpha}$

in $R_2 \cap S$. If $s_\gamma \in \mathcal{H}(s_\alpha s_\beta, s_1)$, then $s_1 s_\alpha s_\beta s_\gamma$ is a 4-hole which is disjoint from the 5-hole $s_2 s_3 s_4 s_5 s_6$ (see Figure 3(a)). Otherwise, $s_\gamma \in \mathcal{H}(s_\alpha s_\beta, s_2)$, and $s_2 s_\alpha s_\beta s_\gamma$ is a 4-hole disjoint from 5-hole $s_1 s_3 s_4 s_5 s_6$.

Case **2:** *There exists* $s_\beta \in R_2 \cap S$ *such that* $s_\alpha s_\beta s_2 s_3 s_4 s_1$ *is a 6-hole.* If $|Cone(ss_1s_3) \cap S| \geq 5$, $Cone(ss_1s_3) \cap S$ contains a 4-hole, since $H(4) = 5$. This 4-hole and the 5-hole $s_1 s_3 s_4 s_5 s_6$ are disjoint (see Figure 3(a)). Otherwise, $|Cone(ss_1s_3) \cap S| \leq 4$, and so $|Cone(s_6 s_1 s_3) \cap S| \geq 5$. This implies that the 4-hole contained $Cone(s_6 s_1 s_3) \cap S$ is disjoint from 5-hole $s_\alpha s_\beta s_2 s_3 s_1$. □

Observation 4 *S is admissible, if $|R_i| = 2$ for some $i \in \{1, 2, \ldots, 12\}$.*

Proof. Without loss of generality assume $|R_2| = 2$. Let $R_2 \cap S = \{s_\alpha, s_\beta\}$, where s_α is the nearest angular neighbor of $\overrightarrow{s_2 s_1}$ in R_2. There are two cases:

Case **1:** s_α *lies inside the triangle* $s_1 s_2 s_\beta$. Let s^*, t^* be as shown in Figure 3(b). If there exists a point $s_\gamma \in S \backslash \{s_\alpha, s_\beta\}$ in the halfplane $\mathcal{H}(s_1 s_2, s_\alpha)$ or $\mathcal{H}(s_1 s_6, s_\alpha)$, then either $s_1 s_\alpha s_\beta s_\gamma$ or $s_2 s_\alpha s_\beta s_\gamma$ is a 4-hole, and the admissibility of S is immediate. Hence, assume that s_α and s_β are the only points of S in these two halfplanes. Observe that $|Cone(ss_1s_3) \cap S| \geq 3$. Since $H(4) = 5$, $Cone(ss_1s_3) \cap S$ contains a 4-hole whenever $|Cone(ss_1s_3) \cap S| \geq 5$. This 4-hole is then disjoint from 5-hole $s_1 s_3 s_4 s_5 s_6$. Therefore, assume that $3 \leq |Cone(ss_1s_3) \cap S| \leq 4$.

Case **1.1:** $|Cone(ss_1s_3) \cap S| = 4$. This implies that $|Cone(s_2 s_3 t^*) \cap S| = 1$. Suppose, $Cone(s_2 s_3 t^*) \cap S = \{s_\gamma\}$. If $s_\gamma \in Cone(s_\beta s_\alpha s_2)$, then $s_\beta s_\alpha s_2 s_\gamma$ is a 4-hole which is disjoint from the 5-hole $s_1 s_3 s_4 s_5 s_6$. Otherwise, $s_2 s_\alpha s_1 s_3 s_\gamma$ is a 5-hole which is disjoint from the 4-hole contained in $Cone(s_3 s_1 s_6) \cap S$, since $|Cone(s_3 s_1 s_6) \cap S| \geq 5$.

Case **1.2:** $|Cone(ss_1s_3) \cap S| = 3$. If $|\mathcal{H}(s_3 s_5, s_4) \cap S| \geq 5$, $H(4) = 5$ immediately implies the admissibility of S. Hence, assume that $|\mathcal{H}(s_3 s_5, s_4) \cap S| = \lambda \leq 4$. Now, since $\overline{\mathcal{H}}(s_1 s_2, s_4) \cap S = \overline{\mathcal{H}}(s_1 s_6, s_4) \cap S = \{s_\alpha, s_\beta\}$ and $Cone(s_2 s_3 t^*) \cap S$ is empty, we have $|\overline{\mathcal{H}}(s_3 s_5, s_4) \cap (R_{10} \cap S)| = 5 - \lambda$. Then there exists $s_\gamma \in \overline{\mathcal{H}}(s_3 s_5, s_4) \cap (R_{10} \cap S)$ such that $s_1 s_2 s_3 s_\gamma s_6$ is a 5-hole which is disjoint from the 4-hole contained in $\mathcal{H}(s_3 s_\gamma, s_4)$.

Case **2:** $s_1, s_2, s_\beta, s_\alpha$ *are in convex position.* Then $s_1 s_3 s_4 s_5 s_6$ and $s_1 s_\alpha s_\beta s_2 s_3$ are two 5-holes sharing the edge $s_1 s_3$. Now, since $|S \backslash \{s_1, s_3\}| = 10$, by the pigeonhole principle either $|\mathcal{H}(s_1 s_3, s_2) \cap S| \geq 5$ or $|\overline{\mathcal{H}}(s_1 s_3, s_2) \cap S| \geq 5$. Therefore, the 4-hole contained in $\mathcal{H}(s_1 s_3, s_2) \cap S$ or $\overline{\mathcal{H}}(s_1 s_3, s_2) \cap S$ is disjoint from the 5-hole $s_1 s_3 s_4 s_5 s_6$ or $s_1 s_\alpha s_\beta s_2 s_3$, respectively. □

Equipped with these three observations we proceed with the proof of Theorem 1. For every point $s_i \in \mathcal{V}(B)$, the diagonal $d := s_i s_{i+3}$ is called a *dividing diagonal* of B. A dividing diagonal d of B is called an $(a, b) -$ *splitter* of S, where $a \leq b$ are integers, if either $|\mathcal{H}(s_i s_{i+3}, s_{i+1}) \cap S \backslash \mathcal{V}(B)| = a$ and $|\overline{\mathcal{H}}(s_i s_{i+3}, s_{i+1}) \cap S \backslash \mathcal{V}(B)| = b$, or $|\overline{\mathcal{H}}(s_i s_{i+3}, s_{i+1}) \cap S \backslash \mathcal{V}(B)| = a$ and $|\mathcal{H}(s_i s_{i+3}, s_{i+1}) \cap S \backslash \mathcal{V}(B)| = b$.

From Observations 3 and 4, we have $|R_i| \leq 1$ for all $i \in \{1, 2, \ldots, 12\}$. Now, if some dividing diagonal of B, say $s_1 s_4$, is a $(0, 6)$-splitter of S with $|\mathcal{H}(s_1 s_4, s_2) \cap S \backslash \mathcal{V}(B)| = 6$, then from Observation 2, $|\overline{\mathcal{H}}(s_2 s_3, s_1) \cap S| \leq 3$, and hence $|R_2| + |R_7| \geq 3$. Then, either $|R_2| \geq 2$ or $|R_7| \geq 2$, and the admissibility of S is immediate from Observations 3 and 4. Therefore, no dividing diagonal of B is a $(0, 6)$-splitter of S. The only cases which remain to be considered are:

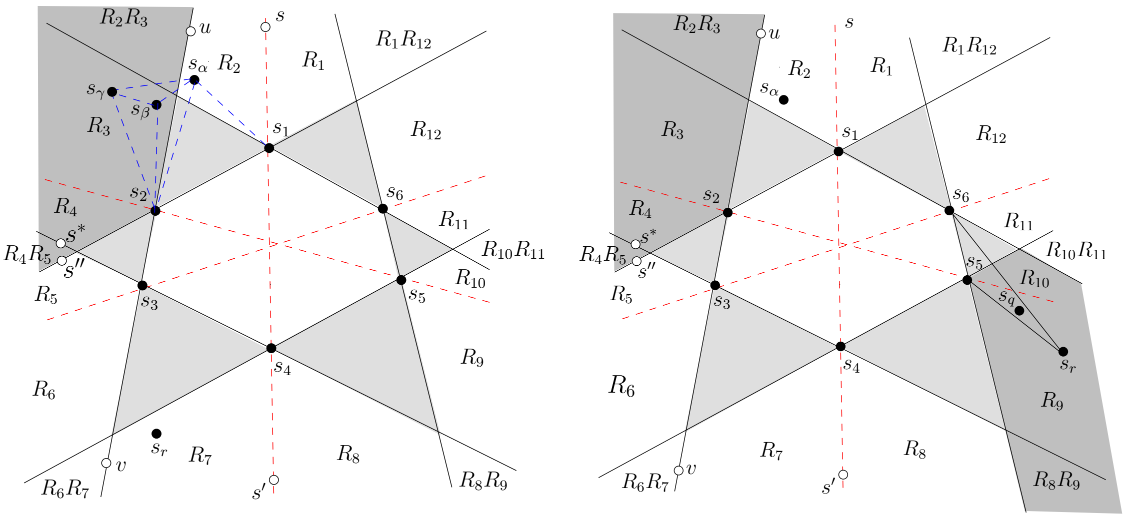
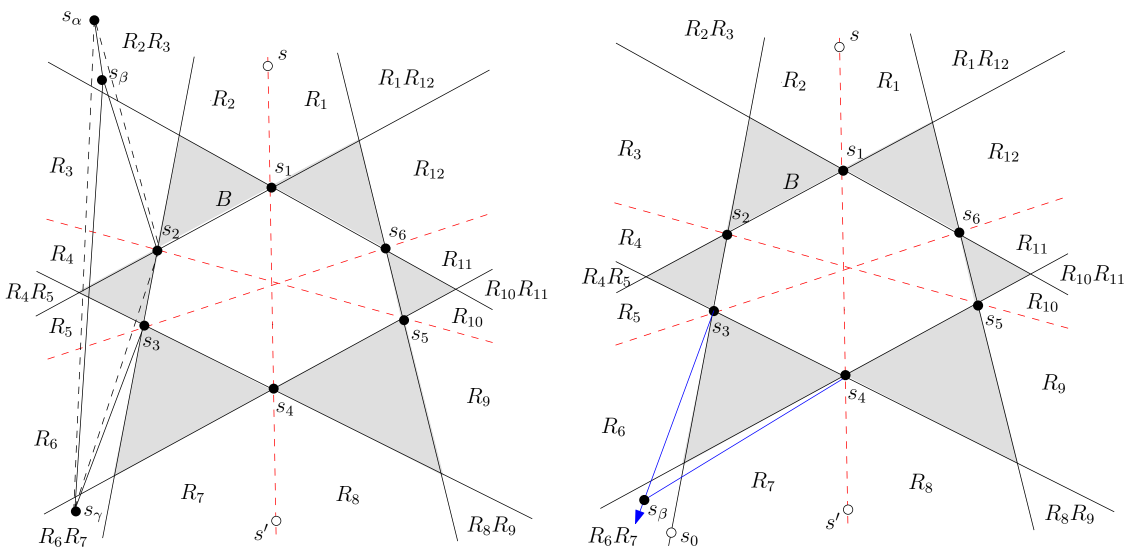


Fig. 4. (a) s_1s_4 is a $(1,5)$-splitter of S such that $|R_2| = |R_7| = 1$, (b) s_1s_4 is a $(2,4)$-splitter of S such that $|R_2| = 1$.

Case **1:** s_1s_4 *is a* $(1,5)$*-splitter of* S *with* $|\mathcal{H}(s_1s_4, s_2) \cap S \setminus \mathcal{V}(B)| = 5$. Observations 2, 3, and 4 imply that $|R_2| = |R_7| = 1$. Let s'', s^*, u, v be as shown in Figure 4(a). Let $s_\alpha \in R_2 \cap S$ and $s_r \in R_7 \cap S$. Now, $|\overline{\mathcal{H}}(s_2s_3, s_1) \cap S| = 3$ and Observation 2 implies that either $|Cone(us_2s'') \cap S| = 2$ or $|Cone(vs_3s^*) \cap S| = 2$. Without loss of generality assume, $|Cone(us_2s'') \cap S| = 2$. Let $s_\beta, s_\gamma \in Cone(us_2s'') \cap S$, such that s_β is the nearest angular neighbor of $\overrightarrow{s_2u}$ in $Cone(us_2s'')$. If $s_\alpha \in \mathcal{I}(s_2s_1s_\beta)$, then either $s_2s_\alpha s_\beta s_\gamma$ or $s_1s_\alpha s_\beta s_\gamma$ is a 4-hole which is disjoint from the 5-hole $s_1s_3s_4s_5s_6$ or $s_2s_3s_4s_5s_6$, respectively. Otherwise, $s_\alpha \notin \mathcal{I}(s_2s_1s_\beta)$ and $s_1s_\alpha s_\beta s_2$ is a 4-hole. It can be assumed that the forbidden zone of $s_1s_\alpha s_\beta s_2$ is empty in $\overline{\mathcal{H}}(s_2s_3, s_1) \cap S$. Then, $s_\beta \in \mathcal{I}(s_2s_\alpha s_\gamma)$ (see Figure 4(a)). If $s_\delta \in S$ is the remaining point in $\mathcal{H}(s_2s_3, s_\gamma) \cap S$, then either $s_\alpha s_\beta s_\gamma s_\delta$ or $s_2s_\beta s_\gamma s_\delta$ is a 4-hole disjoint from the 5-hole $s_1s_3s_4s_5s_6$.

Case **2:** s_1s_4 *is a* $(2,4)$*-splitter of* S *with* $|\mathcal{H}(s_1s_4, s_2) \cap S \setminus \mathcal{V}(B)| = 4$. Without loss of generality, suppose $|R_2| = 1$. Let $s_\alpha \in R_2 \cap S$. Refer to Figure 4(b). If $|R_7| \neq 0$, there exists $s_\beta \in R_7 \cap S$ such that $s_\alpha s_2s_3s_\beta s_4$ is a 5-hole which is disjoint from the 4-hole contained in $(\mathcal{H}(s_1s_4, s_5) \cap S) \cup \{s_1\}$. Therefore, assume that $|R_7| = 0$. Similarly, it can be shown that S is admissible unless $|R_1| = |R_8| = 0$. So, $|\overline{\mathcal{H}}(s_5s_6, s_1) \cap S| = 2$. Let $\overline{\mathcal{H}}(s_5s_6, s_1) \cap S = \{s_q, s_r\}$. If s_5, s_6, s_q, s_r are in convex position, then this 4-hole is disjoint from the 5-hole $s_\alpha s_2s_3s_4s_1$. Therefore, assume that $s_q \in \mathcal{I}(s_5s_6s_r)$. This implies that either $s_q, s_r \in \overline{\mathcal{H}}(s_1s_6, s_4)$ or $s_q, s_r \in \overline{\mathcal{H}}(s_4s_5, s_1)$. If $s_q, s_r \in \overline{\mathcal{H}}(s_1s_6, s_4)$, then $s_\alpha s_6s_qs_r$ is a 4-hole which is disjoint from the 5-hole $s_1s_2s_3s_4s_5$. Therefore, let $s_q, s_r \in \overline{\mathcal{H}}(s_4s_5, s_1) \cap S$ (see Figure 4(b)). Again, S is admissible unless R_6R_7 is empty in S. If $|Cone(ss_1s_2) \cap \overline{\mathcal{H}}(s_2s_3, s_1) \cap S| \geq 3$, then $|\overline{\mathcal{H}}(s_1s_2, s_3) \cap S| \geq 4$, and the admissibility of S follows from Observation 2. Again, if $|Cone(ss_1s_2) \cap \overline{\mathcal{H}}(s_2s_3, s_1) \cap S| \leq 1$, then $|R_5| + |R_6| \geq 2$. From Observations 3 and 4, it suffices to consider $|R_5| = |R_6| = 1$. Then, $\{s_1, s_2, s_3, s_6\} \cup (R_5 \cap S)$ forms a 5-hole which is disjoint from the 4-hole $s_rs_qs_5s_4$. Therefore, assume that $|Cone(ss_1s_2) \cap \overline{\mathcal{H}}(s_2s_3, s_1) \cap S| = 2$. Let $s_\beta, s_\gamma \in Cone(us_2s'') \cap S$, such that s_β is the nearest angular neighbor of $\overrightarrow{s_2u}$ in $Cone(us_2s'')$. Refer to Figure 4(a). If $s_\alpha \in \mathcal{I}(s_2s_1s_\beta)$, then either $s_2s_\alpha s_\beta s_\gamma$ or $s_1s_\alpha s_\beta s_\gamma$ is a 4-hole which is disjoint from the 5-hole $s_1s_3s_4s_5s_6$ or $s_2s_3s_4s_5s_6$, respectively. Otherwise, $s_\alpha \notin \mathcal{I}(s_2s_1s_\beta)$ and $s_1s_\alpha s_\beta s_2$ is a

4-hole. It can be assumed that the forbidden zone of $s_1 s_\alpha s_\beta s_2$ is empty in $\overline{\mathcal{H}}(s_2 s_3, s_1) \cap S$. Then, $s_\beta \in \mathcal{I}(s_2 s_\alpha s_\gamma)$. If $s_\delta \in S$ is the remaining point in $\mathcal{H}(s_2 s_3, s_\gamma) \cap S$, then either $s_\alpha s_\beta s_\gamma s_\delta$ or $s_2 s_\beta s_\gamma s_\delta$ is a 4-hole disjoint from the 5-hole $s_1 s_3 s_4 s_5 s_6$.



(a) (b)

Fig. 5. All the dividing diagonals of B are $(3,3)$-splitters of S: (a) *Case* 3.1, (b) *Case* 3.2

Case **3:** *All the dividing diagonals of the hexagon B are $(3,3)$-splitters of S.* Suppose $R_1 \cap S$ is non-empty. Then, there exists $s_\alpha \in R_1 \cap S$ such that $s_\alpha s_1 s_4 s_5 s_6$ is a 5-hole. This 5-hole is disjoint from the 4-hole contained in $\mathcal{H}(s_1 s_4, s_2) \cap S$, since $H(4) = 5$. Therefore, it can be assumed that $|R_i| = 0$ for $i \in \{1, 2, \ldots, 12\}$. Observation 2 and the fact that $|S \backslash \mathcal{V}(B)| = 6$ now implies that for any point $s_i \in \mathcal{V}(B)$, $|\overline{\mathcal{H}}(s_i s_{i+1}, s_{i+2}) \cap S| = 3$. Therefore, regions in the exterior of the hexagon B, where the regions of the type $R_i R_j$ intersect must be empty in S. Now, we consider the following two cases:

Case **3.1:** $|R_i R_j| \leq 2$ *for all pair of indices* i, j. Observe that either $|R_2 R_3| + |R_4 R_5| \geq 2$ or $|R_4 R_5| + |R_6 R_7| \geq 2$. Without loss of generality assume that $|R_2 R_3| + |R_4 R_5| \geq 2$. To begin with, suppose that $|R_2 R_3| + |R_4 R_5| = 2$, with $(R_2 R_3 \cup R_4 R_5) \cap S = \{s_\alpha, s_\beta\}$ and $R_6 R_7 \cap S = \{s_\gamma\}$. If the four points $s_2, s_\alpha, s_\beta, s_\gamma$ form a convex quadrilateral, S is clearly admissible. Otherwise, let $s_\beta \in \mathcal{I}(s_2 s_\alpha s_\gamma)$ and $R_1 R_{12} \cap S = \{s_\delta\}$. Depending on the position of the point s_δ, either $s_2 s_\beta s_\alpha s_\delta$ or $s_\delta s_\alpha s_\beta s_\gamma$ is a 4-hole which is disjoint from the 5-hole $s_1 s_3 s_4 s_5 s_6$ (see Figure 5(a)). Now, let $|R_2 R_3| + |R_4 R_5| = 3$ and without loss of generality, assume $|R_2 R_3| = 2$ and $|R_4 R_5| = 1$. Then $|R_1 R_{12}| = 0$ and $|R_{10} R_{11}| = 1$, since all three dividing diagonals of B are $(3,3)-$splitters of S. From symmetry, it is the same case as before.

Case **3.2:** $|R_i R_j| = 3$ *some pair of indices* i, j. Without loss of generality assume that $|R_1 R_{12}| = 3$. This implies that $|R_6 R_7| = 3$. Let s_0 be as shown in Figure 5(b) and $s_\alpha, s_\beta, s_\gamma$ be the first, second, and third angular neighbors of $\overrightarrow{s_3 s_0}$ in $R_6 R_7$, respectively. If $s_\alpha \in \mathcal{I}(s_3 s_4 s_\beta)$ the admissibility of S follows from the fact that $s_3 s_\alpha s_\beta s_\gamma$ or $s_4 s_\alpha s_\beta s_\gamma$ is a 4-hole which is disjoint from the 5-hole $s_1 s_2 s_4 s_5 s_6$ or $s_1 s_2 s_3 s_5 s_6$, respectively. Otherwise, $s_\alpha \notin \mathcal{I}(s_3 s_4 s_\beta)$, and similarly $s_\gamma \notin \mathcal{I}(s_2 s_3 s_\beta)$.

Then, either $s_1 s_3 s_\beta s_\alpha s_4$ or $s_1 s_3 s_\beta s_\gamma s_2$ is a 5-hole which is disjoint from the 4-hole contained in $(R_1 R_{12} \cap S) \cup \{s_5, s_6\}$. □

4 Remarks and Conclusions

In this paper we proved that $H(4, 5) = 12$, that is, every set of 12 points in plane in general position contains a 4-hole and a disjoint 5-hole, thus improving a result of Hosono and Urabe [12]. The proof uses a Ramsey type result for 11 points proposed by Aichholzer et al. [1].

The most important case that remains to be settled is that of $H(5, 5)$. Urabe and Hosono [13] proved that $16 \leq H(5, 5) \leq 20$, and later improved the lower bound to 17 [12]. There is still a substantial gap between the upper and lower bounds of $H(5, 5)$. We believe that a new Ramsey-type result similar to Theorem 2 might be useful in obtaining better bounds on $H(5, 5)$.

However, we are still far from establishing non-trivial bounds on $H(6, l)$, for $0 \leq l \leq 6$, since the exact value of $H(6) = H(6, 0)$ is still unknown. The best known bounds are, $30 \leq H(6) \leq ES(9) \leq 1717$. The lower bound is due to Overmars [18] and the upper bound due to Gerken [9].

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