The Discrete Voronoi Game in $\mathbb{R}^2$*

Aritra Banik† Bhaswar B. Bhattacharya‡ Sandip Das§ Satyaki Mukherjee¶

Abstract

In this paper we study the last round of the discrete Voronoi game in $\mathbb{R}^2$, a problem which is also of independent interest in competitive facility location. The game consists of two players P1 and P2, and a finite set U of users in the plane. The players have already placed two disjoint sets of facilities F and S, respectively, in the plane. The game begins with P1 placing a new facility followed by P2 placing another facility, and the objective of both the players is to maximize their own total payoffs. In this paper we propose polynomial time algorithms for determining the optimal strategies of both the players for arbitrarily located existing facilities F and S. We show that in the L_1 and the L_∞ metrics, the optimal strategy of P2, given any placement of P1, can be found in $O(n \log n)$ time, and the optimal strategy of P1 can be found in $O(n^5 \log n)$ time. In the L_2 metric, the optimal strategies of P2 and P1 can be obtained in $O(n^2)$ and $O(n^8)$ times, respectively.

1 Introduction

The main objective in any facility location problem is to place a set of facilities serving a set of users such that certain optimality criteria are satisfied. Facilities and users are generally modeled as points in the plane. The set of users (demands) is either *discrete*, consisting of finitely many points, or *continuous*, that is, a region where every point is considered to be a user. We assume that the facilities are equally equipped in all respects, and a user always avails the service from its nearest facility. Consequently, each facility has its *service zone*, consisting of the set of users that are served by it. (Refer to the book by Drezner and Hamacher [14] for a comprehensive discussion on facility location problems and their manifold many generalizations.)

Competitive facility location is concerned with the favorable placement of facilities by competing market players [16, 17]. In general, the users choose the facilities based on the nearest-neighbor rule, and the optimization criteria is to maximize the cardinality or the area of the service zone depending on whether the demand region is discrete or continuous, respectively. For a recent survey on the applications of competitive facility location in economics and operations research, refer to [13].

*A preliminary version of this paper appeared in the *Proc. 25th Canadian Conference on Computational Geometry*, Ontario, Canada, 2013.

†Department of Computer Science and Engineering, Indian Institute of Technology, Jodhpur, India, aritra@iitj.ac.in

‡Department of Statistics, University of Pennsylvania, Philadelphia, USA, bhaswar@wharton.upenn.edu

§Advanced Computing and Microelectronics Unit, Indian Statistical Institute, Kolkata, India, sandipdas@isical.ac.in

¶Department of Mathematics, University of California, Berkeley, USA, satyaki@berkeley.edu

In this paper, we study the *discrete Voronoi game in* $\mathbb{R}^2$ *in the presence of existing facilities*, which is the game-theoretic variant of a competitive facility location problem for discrete demands in $\mathbb{R}^2$. The game consists of two players P1 and P2, and a finite set U of users in the plane. The players have already placed two sets of facilities F and S, respectively, in the plane. To begin with, P1 places a new facility followed by P2 placing another facility, and the objective of both the players is to maximize their own total payoffs, where the payoff of P1/P2 is the cardinality of the set of points in U which are closer to a facility owned by P1/P2 than to every facility owned by P2/P1. Apart from being the $(m+1)$-th round of discrete Voronoi game in $\mathbb{R}^2$, when $|F| = |S| = m$, this problem is also of independent interest in competitive facility location: Imagine two competing companies are providing service to a set of users in a city. Suppose both these companies already have their respective service centers located in different parts of the city. If each of them now wishes to open a new service center while attempting to maximize its total payoff, then the problem is an instance of the Voronoi game described above.

In this paper, we develop algorithms and provide geometric characterizations for the optimal strategies of the two players for the discrete Voronoi game in $\mathbb{R}^2$ in the presence of existing facilities.

1.1 Related Work The rich history of competitive facility location problems goes back to the 1929 seminal paper by Hotelling [20] that considers the competitive facility location problem where the users are located uniformly on a line segment. Dehne et al. [12] studied a competitive facility location problem for continuous demand regions, where the problem is to find a new point q amidst a set of n existing points $\mathcal{F}$ such that the Voronoi region of q is maximized. They showed that when the points in $\mathcal{F}$ are in convex position, the area function has only a single local maximum inside the region where the set of Voronoi neighbors do not change. For the same problem, Cheong et al. [10] gave a near-linear time algorithm that determines the location of the new optimal point approximately, when the points in $\mathcal{F}$ are in general position. A variation of this problem, involving maximization of the area of Voronoi regions of a set of points placed inside a circle, was considered by Bhattacharya [7]. In the discrete user case, the analogous problem is to place a set of new facilities amidst a set of existing ones such that the number of users served by the new facilities is maximized [8, 9].

A game-theoretic analogue of such competitive problems for continuous demand regions is a situation where two players alternately place two disjoint set of facilities in the demand region. In this case, the payoff of player P1/P2 is the area of the region that is closer to the facilities owned by P1/P2 than to the other player, and the player which finally owns the larger area is the winner of the game. Ahn et al. [1] studied a one-dimensional Voronoi game, where the demand region is a line segment. They showed that when the players place one facility each for m rounds, the second player always has a winning strategy that guarantees a payoff of $1/2 + \varepsilon$, with $\varepsilon > 0$. However, the first player can force ε to be arbitrarily small. On the other hand, in the one-round game, where the players alternately place m facilities simultaneously, the first player always has a winning strategy. The one-round Voronoi game in $\mathbb{R}^2$ was studied by Cheong et al. [11], for a square-shaped demand region. They proved that for any placement W of the first player, with $|W| = m$, there is a placement B of the second player $|B| = m$ such that the payoff of the second player is at least $1/2 + \alpha$, where $\alpha > 0$ is an absolute constant and m large enough. Fekete and Meijer [18] studied the two-dimensional one-round game played on a rectangular demand region with aspect ratio ρ. Recently, variants of these games when the demand region is a graph equipped with the shortest-path distance [3], and the demand region is a simple polygon equipped with the geodesic distance [6], have been studied.

A natural variant of this game can be played on a graph equipped with the shortest-path distance. As before, the players alternately chose nodes (facilities) from the graph, and all vertices (customers) are then assigned to their closest facilities based on the graph distance. The payoff of a player is the number of customers assigned to it. Dürr and Thang [15] showed that deciding the existence of a Nash equilibrium for a given graph is NP-hard. Teramoto et al. [25] studied the same problem in a restricted case: the game arena is an arbitrary graph, the first player occupies just one vertex which is predetermined, and the second player occupies m vertices in any way. They proved that even in this strongly case it is NP-hard to decide whether the second player has a winning strategy. They also proved that for a given graph G and the number r of rounds, determining whether the first player has a winning strategy on G is PSPACE-complete. Recently, Gerbner et al. [19] derived bounds on the payoff of the players for many graphs, and showed that there are graphs for which the second player gets almost all vertices.

Banik et al. [4] introduced the discrete Voronoi game in $\mathbb{R}$, where the demand region is a finite set of points on the line, and users avail the services of facilities closest to them, in Euclidean distance. Given a set of users $U \subset \mathbb{R}$, with $|U| = n$, player Player 1 (P1) chooses a set of m facilities, following which Player 2 (P2) chooses another disjoint set of m facilities, and the objective of both the players is to maximize their respective payoffs. The authors showed that if the sorted order of the points in U along the line is known, then the optimal strategy of P2, given any placement of facilities by P1, can be computed in $O(n)$ time, and the optimal strategy of P1 can be computed in $O(n^{m-\lambda_m})$ time, where $0 < \lambda_m < 1$, is a constant depending only on m. Recently, using connections to ϵ-nets, Banik et al. [5] obtained approximation algorithms for a version of the discrete Voronoi game in $\mathbb{R}^2$.

1.2 Summary of Results In this paper we initiate the study of the discrete Voronoi game when the users are a finite set of points in $\mathbb{R}^2$, equipped with the L_1, L_2, or L_∞ metrics. To this end, for a finite set U of users and a set $\mathcal{F}$ of facilities, define for every $f \in \mathcal{F}$,

$$U(f, \mathcal{F}) = \{u_a \in U : d(u_a, f) < d(u_a, h), \forall h \in \mathcal{F} \backslash \{f\}\}, \tag{1.1}$$

where the distance $d(\cdot,\cdot)$ is measured in the L_1, L_2, or L_∞ metrics (to be denoted by $d_1(\cdot,\cdot)$, $d_2(\cdot,\cdot)$, and $d_\infty(\cdot,\cdot)$, respectively).

Now, consider a set U of users in the plane and two players P1 and P2. Throughout the paper, we assume that two facilities are not allowed to be placed in the same location. For any placement of facilities A and B by P1 and P2, respectively, the payoff of P2, to be denoted by $\mathcal{P}_2(A, B)$, is defined as the cardinality of the set of points in U which are closer to a facility owned by P2 than to every facility owned by P1, that is, $\mathcal{P}_2(A, B) = |\bigcup_{f \in B} U(f, A \bigcup B)|$. Similarly, the payoff of P1, $\mathcal{P}_1(A, B) = |U| - \mathcal{P}_2(A, B)$. Note that this definition implies that if an user is equidistant from a facility in A and another facility in B, then it contributes to the payoff of P1, that is, ties are broken in favor of P1.

The problem studied in this paper can now be formally stated as follows:

One Round Discrete Voronoi Game in $\mathbb{R}^2$ in Presence of Existing Facilities: Let U be a set of n users in the plane, and F and S be two sets of facilities owned by two competing players P1 and P2, respectively. To begin with, P1 chooses a facility f_1 following which P2 chooses another facility f_2 such that *(a)* $\max_{f_2' \in \mathbb{R}^2} \mathcal{P}_2(F \bigcup \{f_1\}, S \bigcup \{f_2'\})$ is attained at the point f_2.

(b) $\max_{f \in \mathbb{R}^2} \nu(f)$ is attained at the point f_1, where

$$\nu(f) = n - \max_{f_2' \in \mathbb{R}^2} \mathcal{P}_2(F \bigcup \{f\}, S \bigcup \{f_2\}). \tag{1.2}$$

The quantity $\nu(f_1)$ is called the optimal payoff of P1 and f_1 is the optimal strategy of P1. Hereafter, we shall refer to this game as $G_n(F, S)$.[1]

When $|F| = |S| = m$, the optimal strategies of the game $G_n(F, S)$ is the last round of the $(m+1)$-round discrete Voronoi game in $\mathbb{R}^2$. To the best of our knowledge, the $G_n(F, S)$ game has never been studied before in the generality described above. However, few special cases are known. For example, when both F and S are empty, then it is a well-known fact that optimal strategy of P1 in the $G_n(F, S)$ game is at the halfspace median of U [22], which can be computed in $O(n \log^3 n)$ time [21]. However, when the sets F and S are non-empty the problem becomes much more complicated, and answering questions regarding the strategy of P1 is often very difficult. In this paper, we initiate the study of this game, for general placements of the existing facilities F and S, and propose polynomial time algorithms for the optimal strategies of both the players and provide geometric characterizations of the solution space.

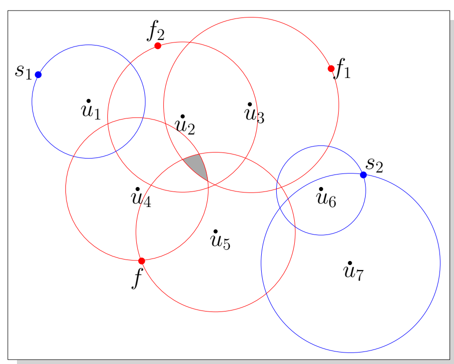


Figure 1: Optimal strategy of P2 in the L_2 metric: The users served by $F \bigcup \{f\}$ correspond to the centers of the red circles (the users u_2, u_3, u_4, and u_5). The centers of the blue circles (the users u_1, u_6, and u_7) correspond to the users served by S. The optimal strategy of P2 is to place a new facility in the region intersected by the maximum number of red circles (the shaded region).

We begin with the optimal strategy of P2. It is easy to see that the optimal strategy of P2, given any placement of P1, follows from the results of Cabello et al. [9]. Suppose we are given a set of users U, existing facilities F and S, and any placement of a new facility f by P1. Let $U_1 \subseteq U$ denote the subset of users that are served by P1, in presence of F, S, and f. For every point $u \in U_1$, consider the *nearest-facility disk* C_u centered at u and passing through the facility in $F \bigcup \{f\}$ which is closest to u. Note that a new facility s placed by P2 will serve any user $u \in U_1$ if and only if $s \in C_u$. If $\mathcal{C} = \{C_u | u \in U_1\}$, the optimal strategy for P2, given any placement f of P1, is to place the new facility at a point where maximum number of disks in $\mathcal{C}$ overlap (Figure 1).

[1] A related problem was considered by Plastria [24], where the goal was to place a new facility among a set of existing friendly facilities F and competitive facilities C, with the objective to maximize the total payoff of the friendly facilities, while taking in to account the possibility that the competing facilities can raise their current quality in order to gain back users.

Therefore, in the L_2 metric, this is the problem of finding the maximum depth in an arrangement of n disks, and can be computed in $O(n^2)$ time [2]. In the L_1/L_∞ metric this becomes the problem of finding the maximum depth in an arrangement of squares, which can be done in $O(n \log n)$ time [2].

To obtain the optimal strategy of P1 each cell in the arrangement of the nearest-facility squares/discs have to partitioned further into finer cells inside which the payoff of P1 remains fixed. To this end, we have the following theorems:

Theorem 1.1. *In the L_1 and L_∞ metrics the optimal strategy of P1 in the $G_n(F, S)$ game can be found in $O(n^5 \log n)$ time.*

Theorem 1.2. *In the L_2 metric the optimal strategy of P1 in the $G_n(F, S)$ game can be found in $O(n^8)$ time.*

The above theorems achieve more than just the optimal strategy of P1. In fact, our algorithm computes the locus of all points which attains the maximum payoff of P1. More generally, it computes the level sets $\mathscr{L}(r) = \{f \in \mathbb{R}^2 : \nu(f) \geq r\}$, with $\nu(f)$ as in (1.2). Note that if f_1 is an optimal location of P1, then $\mathscr{L}(\nu(f_1))$ is the set of all points which maximizes the payoff of P1.

2 Preliminaries

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of n users in the plane and F and S be the sets of existing facilities of two competing players P1 and P2, respectively. The set of facilities F and S, will divide the set of users U into two groups U_F and U_S, where U_F is the set of users served by the facilities placed by P1 and U_S is the set of users served by the facilities placed by P2. Let f be any new placement by P1. Denote by $U_{FS}(f)$, the set of users that are served by f, that is,

$$U_{FS}(f) = \{u_a \in U : d(u_a, f) \leq d(u_a, h), \forall h \in F \bigcup S)\}, \tag{2.1}$$

where the distance $d(\cdot, \cdot)$ is measured in the L_1, L_2, or L_∞ metrics. The set of users that are served by the set of facilities F and S after the placement of f, will be denoted by $U_{F\backslash f}$ and $U_{S\backslash f}$ respectively. More formally,

$$U_{F\backslash f} = \bigcup_{h \in F} U(h, F \bigcup S \bigcup \{f\}) \text{ and } U_{S\backslash f} = \bigcup_{h \in S} U(h, F \bigcup S \bigcup \{f\}). \tag{2.2}$$

Hence, any facility f by P1 will divide the set of users into three disjoint sets $U_{FS}(f)$, $U_{F\backslash f}$ and $U_{S\backslash f}$ (see Figure 2(a)). Now, any new placement s by P2 can serve a subset of users from all these three sets. Let $U_f(s) \subseteq U_{FS}(f)$ be the subset of users that s steals from f, that is,

$$U_f(s) = \{u_a \in U_{FS}(f) | d(u_a, s) < d(u_a, f)\}.$$

Similarly, define the set of users

$$U_{F\backslash f}(s) = \{u_a \in U_{F\backslash f} | d(u_a, s) < d(u_a, f_k), \ \forall f_k \in F\} \tag{2.3}$$

(see Figure 2(b)). This is the subset of users that s steals from $F \backslash \{f\}$.

(a) (b)

Figure 2: Distribution of users among facilities in F and S: (a) after placing f, (b) after placing both f and s.

Observe for any placement f and s by P1 and P2 respectively, the payoff of P2 is

$$\mathcal{P}_2(F \bigcup \{f\}, S \bigcup \{s\}) = |U_{S\setminus f}| + |U_f(s)| + |U_{F\setminus f}(s)|$$

Note that given any placement of the facility f, $U_{S\setminus f}$ does not depend on s. Thus, for any given placement of facility f by P1, optimal placement by P2 corresponds to the point $s \in \mathbb{R}^2$ which maximizes $|U_f(s)| + |U_{F\setminus f}(s)|$. For any placement of facility f by P1 define *effective depth* of f, denoted by $\delta(f)$, as

$$\delta(f) = \max_{s\in\mathbb{R}^2}(|U_{S\setminus f}| + |U_f(s)| + |U_{F\setminus f}(s)|) = |U_{S\setminus f}| + \max_{s\in\mathbb{R}^2}(|U_f(s)| + |U_{F\setminus f}(s)|). \tag{2.4}$$

The optimal strategy of P1 is to find the point f with the minimum effective depth.

3 Optimal Placement of P1 in the L_1 metric

In this section we consider the optimal strategy of P1 in the L_1 metric. The analogous problem in the L_∞ metric can be dealt with similarly by rotating the axes by 45°.

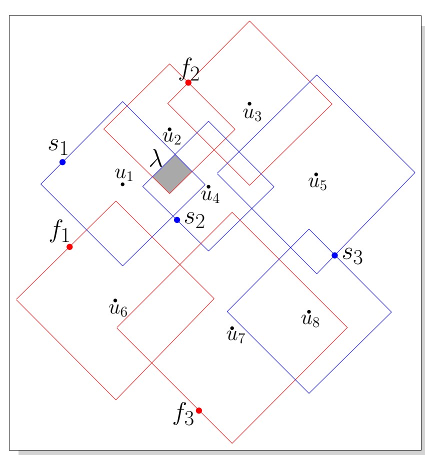


Figure 3: Distribution of users among the facilities F and S.

For any two points $x, y \in \mathbb{R}^2$, denote by $d_1(x, y)$ the L_1 distance between x and y. For any user $u_a \in U$, let R_a be the open L_1 ball centered at u_a with radius $d_1(u_a, h)$, where $h \in F \bigcup S$ is the facility closest to u_a (in L_1 distance) among the set of facilities $F \bigcup S$. Note that the boundary of R_a is a square centered at u_a and diagonals coinciding with the x-axis and y-axis and one of the sides touching h. Let $\mathcal{R}_{FS} = \{R_a | u_a \in U\}$ be the collection of these nearest-facility squares, which will tessellate $\mathbb{R}^2$ into a set of regions (see Figure 3). Denote this tessellation by $\mathcal{T}(F, S)$.

For each cell λ in this tessellation $\mathcal{T}(F, S)$, and any placement of a facility f by P1 in that cell, the sets $U_{S\setminus f}$, $U_{FS}(f)$ and $U_{F\setminus f}$ remain unchanged. For notational brevity, for any fixed cell $\lambda \in \mathcal{T}(F, S)$ and for all points $f \in \lambda$, the three sets $U_{S\setminus f}$, $U_{FS}(f)$ and $U_{F\setminus f}$ will be denoted by $U_{S\setminus\lambda}$, $U_{FS}(\lambda)$ and $U_{F\setminus\lambda}$, respectively. For example, in Figure 3 for the cell λ,

$$U_{S\setminus\lambda} = \{u_5, u_8\}, \quad U_{FS}(\lambda) = \{u_1, u_2, u_4\}, \quad U_{F\setminus\lambda} = \{u_3, u_6, u_7\}.$$

Our goal is to tessellate λ further such that for all points in each finer the effective depth remains fixed. To this end, for any placement of facility x by P1 and for any user u_a, let $R_a(x)$ be the open square centered at u_a and passing through the facility closest (in L_1 distance) to u_a from the set of facilities $F \bigcup S \bigcup \{x\}$. Recall that $\delta(x)$ denotes the effective depth of x (2.4). The following lemma shows that the effective depth of a point $x \in \mathbb{R}^2$ can be determined by the intersection properties of pairwise nearest-facility squares.

Lemma 3.1. *If points x, y belong to the same cell λ of $\mathcal{T}(F, S)$ with $\delta(x) \neq \delta(y)$, then there exist two users $u_a, u_b \in U_{FS}(\lambda) \bigcup U_{F\setminus\lambda}$ such that $R_a(x) \bigcap R_b(x) \neq \varnothing$ and $R_a(y) \bigcap R_b(y) = \varnothing$ or vice versa.*

Proof. For any placement x in the cell λ of $\mathcal{T}(F, S)$, recall that $\delta(x)$ is maximum depth of the collection $\mathcal{R}(x) := \{R_a(x) : u_a \in U_{FS}(\lambda) \bigcup U_{F\setminus\lambda}\}$ of squares. Let $\mathcal{P}(x)$ be the subset of $\mathcal{R}(x)$ which attains this maximum depth, that is, the largest subset of $\mathcal{R}(x)$ for which $\bigcap_{R \in \mathcal{P}(x)} R \neq \varnothing$.

Now, suppose there are two points $x, y \in \lambda$ such that, $\delta(x) \neq \delta(y)$, and for all $u_a, u_b \in U_{FS}(\lambda) \bigcup U_{F\setminus\lambda}$, $R_a(x) \bigcap R_b(x) \neq \varnothing$ if and only if $R_a(y) \bigcap R_b(y) \neq \varnothing$. Without loss of generality, assume $\delta(x) > \delta(y)$.

By definition, for each pair $R_a(x), R_b(x) \in \mathcal{P}(x)$, $R_a(x) \bigcap R_b(x) \neq \varnothing$. Therefore, by assumption, $R_a(y) \bigcap R_b(y) \neq \varnothing$, for each pair $R_a(y), R_b(y) \in \mathcal{P}(y) := \{R_a(y) : R_a(x) \in \mathcal{P}(x)\}$. Therefore, by Helly's theorem for axis-parallel squares [23, Corollary 1.5], $\bigcap_{R \in \mathcal{P}(y)} R \neq \varnothing$, which implies that

$$\delta(y) \geq |\mathcal{P}(y)| = |\mathcal{P}(x)| = \delta(x),$$

which is a contradiction. This completes the proof of the result. □

In light of this lemma we define, for each pair of users $u_a, u_b \in U$, and any placement of facility $x \in \mathbb{R}^2$ by P1, the indicator variable,

$$T_{ab}(x) = \begin{cases} 1 & \text{if } R_a(x) \bigcap R_b(x) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

Let $\boldsymbol{T}(x) = ((T_{ab}(x)))_{1 \leq a,b \leq |U|}$ be the 2 dimensional array of size $|U| \times |U|$ where each entry $T_{ab}(x)$ is defined as above. From Lemma 3.1 and the above definition, the following observation is immediate.

Observation 3.1. *If the points x, y belong to the same cell of $\mathcal{T}(F, S)$ and the two arrays $\boldsymbol{T}(x) = \boldsymbol{T}(y)$, in every coordinate, then $\delta(x) = \delta(y)$.*

Hence, the goal is to tessellate cells of $\mathcal{T}(F,S)$ into finer set of cells such that for any two points x and y in the same cell, $\boldsymbol{T}(x) = \boldsymbol{T}(y)$. Observation 3.1 would then imply that for all points in a finer cell, the effective depth remains constant. Hence, by checking each cell once we can find out the point with minimum effective depth. To this end, let $u_a, u_b \in U_{FS}(\lambda) \bigcup U_{F\setminus\lambda}$, and consider

$$\mathcal{T}(u_a, u_b) = \left\{ f \in \mathbb{R}^2 : R_a(f) \bigcap R_b(f) = \varnothing \right\}. \tag{3.1}$$

Now, denote by $f(u_a)$ the facility in $F \bigcup \{f\}$ closest to $u_a \in U_{FS}(\lambda) \bigcup U_{F\setminus\lambda}$. Then

$$\mathcal{T}(u_a, u_b) = \mathcal{T}_1(u_a, u_b) \bigcup \mathcal{T}_2(u_a, u_b) \bigcup \mathcal{T}_2(u_b, u_a) \bigcup \mathcal{T}_3(u_a, u_b),$$

where

$$\begin{aligned}
\mathcal{T}_1(u_a, u_b) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \bigcap R_b(f) = \varnothing \text{ and } f(u_a), f(u_b) \in F \right\}, \\
\mathcal{T}_2(u_a, u_b) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \bigcap R_b(f) = \varnothing \text{ and } f(u_a) \in F,\ f(u_b) = f \right\}, \\
\mathcal{T}_2(u_b, u_a) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \bigcap R_b(f) = \varnothing \text{ and } f(u_b) \in F,\ f(u_a) = f \right\}, \\
\mathcal{T}_3(u_a, u_b) &= \left\{ f \in \mathbb{R}^2 : R_a(f) \bigcap R_b(f) = \varnothing \text{ and } f(u_a) = f(u_b) = f \right\}.
\end{aligned} \tag{3.2}$$

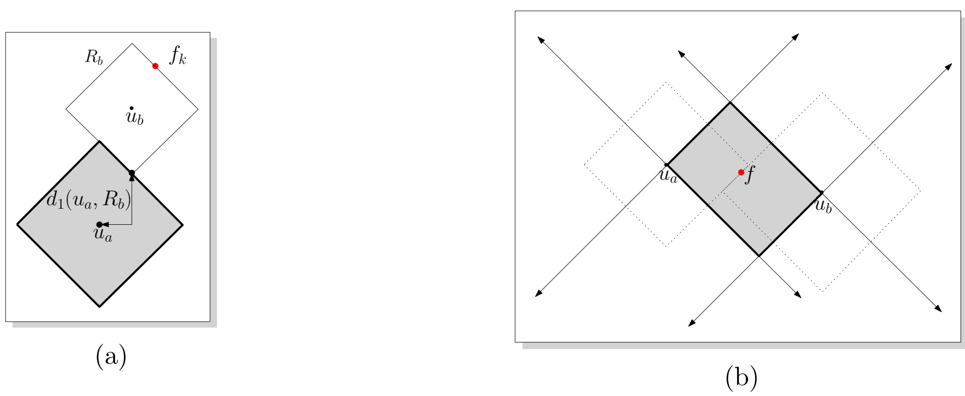


Figure 4: (a) The shaded region denotes the set $\mathcal{T}_2(u_a, u_b)$. (b) The shaded region denotes the set $\mathcal{T}_3(u_a, u_b)$.

Lemma 3.2. *For every pair of users $u_a, u_b \in U_{FS}(\lambda) \bigcup U_{F\setminus\lambda}$, the sets $\mathcal{T}_1(u_a, u_b)$, $\mathcal{T}_2(u_a, u_b)$, $\mathcal{T}_2(u_b, u_a)$, and $\mathcal{T}_3(u_a, u_b)$ can be found in $O(1)$ time.*

Proof. Recall that for $u_a \in U$, the square R_a is the open L_1 ball centered at u_a with radius $d_1(u_a, h)$, where $h \in F \bigcup S$ is the facility closest to u_a (in L_1 distance) among the set of facilities $F \bigcup S$. Then it is easy to see that $\mathcal{T}_1(u_a, u_b) = (R_a \bigcup R_b)^c$.

To compute $\mathcal{T}_2(u_a, u_b)$, without loss of generality, assume that $f(u_b) = f_k \in F$. Let the minimum L_1 distance between u_a and R_b be $d := d_1(u_a, R_b)$ (see Figure 4(a)). Denote by $B_1(u_a, d)$ the closed L_1 ball with center at u_a and distance d. Observe that for any new facility $f \in B_1(u_a, d)$, $R_a(f) \bigcap R_b(f) = \varnothing$, which implies $\mathcal{T}_2(u_a, u_b) = B_1(u_a, d) \cap R_b \setminus R_a$. The set $\mathcal{T}_2(u_b, u_a)$ can be obtained similarly. Both these sets can be computed in $O(1)$ time.

Next, we consider the set $\mathcal{T}_3(u_a, u_b)$. Consider the four lines making 45° and 135° angle with the x-axis and passing through the points u_a and u_b (see Figure 4(b)). These four lines will divide the plane into nine regions, one bounded and the other eight are unbounded. Observe that except for the bounded region, for any facility f placed in one of the unbounded regions, $R_a(f) \bigcap R_b(f)$ will either contain u_a or u_b, that is, $R_a(f) \bigcap R_b(f) \neq \varnothing$. On the other hand, for any point f in the bounded region, the closures of $R_a(f)$ and $R_b(f)$ will share a common edge, but their interiors will not intersect. Therefore, $\mathcal{T}_3(u_a, u_b)$ is the shaded region in Figure 4(b) intersected with $R_a \bigcap R_b$. This can be computed in $O(1)$ time as well. □

***Completing the Proof of Theorem* 1.1**: Consider the arrangement in $\mathbb{R}^2$ generated by the collection of sets

$$\left\{\mathcal{T}(u_a, u_b) : u_a, u_b \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda} \text{ and } \lambda \in \mathcal{T}(F, S)\right\}. \tag{3.3}$$

Note that this arrangement consists of $O(n^2)$ rectangles which can be computed in $O(n^2)$ time by Lemma 3.2 above. These $O(n^2)$ rectangles intersect to generate $O(n^4)$ cells, and by Observation 3.1, all points in a particular cell have the same the effective depth. Note that the effective depth of a cell can be computed in $O(n \log n)$ time, by finding the maximum depth of the arrangement of squares $\{R_a(f) : u_a \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda}\}$, for any point f in the cell [9, Theorem 8]. Therefore, the optimal strategy of P1 in the L_1 metric can be computed in $O(n^5 \log n)$ time. Note that since we search over all cells of the tesselation (3.3), the level sets $L(r) = \{f \in \mathbb{R}^2 : \nu(f) \geq r\}$, where $\nu(f)$ is the payoff P1 when placed at the point $f \in \mathbb{R}^2$ (1.2), can be computed in same running time.

4 Optimal Placement of P1 in the L_2 metric

For $u_a \in U$, denote by C_a the disc centered at u_a and passing through the facility closest to u_a among the set of facilities $F \bigcup S$. Let $\mathcal{C}_{FS} = \{C_a : u_a \in U\}$ be the collection of all such discs, which will tessellate $\mathbb{R}^2$ into a set of regions (see Figure 1). As in the case of the L_1 metric, denote this tessellation by $\mathcal{T}(F, S)$.

For any placement of facility $x \in \mathbb{R}^2$ by P1 and any user $u_a \in U$, let $C_a(x)$ be the open disc centered at u_a and passing through the facility closest to u_a among the facilities in $F \bigcup S \bigcup \{x\}$. As before, for any fixed cell $\lambda \in \mathcal{T}(F, S)$ and for all points $f \in \lambda$, the three sets $U_{S\backslash f}$, $U_{FS}(f)$ and $U_{F\backslash f}$ will be denoted by $U_{S\backslash\lambda}$, $U_{FS}(\lambda)$ and $U_{F\backslash\lambda}$, respectively. Then, similar to Lemma 3.1, we have the following:

Lemma 4.1. *If the points x, y belong to some cell λ of $\mathcal{T}(F, S)$ with $\delta(x) \neq \delta(y)$, then there exist three users $u_a, u_b, u_c \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda}$ such that either $C_a(x) \bigcap C_b(x) \bigcap C_c(x) \neq \varnothing$ and $C_a(y) \bigcap C_b(y) \bigcap C_c(y) = \varnothing$ or vice versa.*

Proof. As in Lemma 3.1, we shall prove the result by contradiction. To this end, assume $\delta(x) > \delta(y)$, and for every three users $u_a, u_b, u_c \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda}$, if $C_a(x) \bigcap C_b(x) \bigcap C_c(x) \neq \varnothing$ then $C_a(y) \bigcap C_b(y) \bigcap C_c(y) \neq \varnothing$.

For any placement x in the cell λ, $\delta(x)$ is maximum depth of the collection $\mathcal{C}(x) := \{C_a(x) : u_a \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda}\}$ of squares. Let $\mathcal{D}(x)$ be the subset of $\mathcal{C}(x)$ which attains this maximum depth, that is, the largest subset of $\mathcal{C}(x)$ for which $\bigcap_{C \in \mathcal{D}(x)} C \neq \varnothing$. By definition, for each triple $C_a(x), C_b(x), C_c(x) \in \mathcal{D}(x)$, $C_a(x) \bigcap C_b(x) \bigcap C_c(x) \neq \varnothing$. Therefore, by assumption,

$C_a(y)\bigcap C_b(y)\bigcap C_c(y) \neq \varnothing$, for each triple $C_a(y), C_b(y), C_b(c) \in \mathcal{D}(y) := \{C_a(y) : C_a(x) \in \mathcal{D}(x)\}$. Therefore, by the Helly's theorem for discs [22, Theorem 1.3.2], $\bigcap_{C\in\mathcal{D}(y)} C \neq \varnothing$, which implies that

$$\delta(y) \geq |\mathcal{D}(y)| = |\mathcal{D}(x)| = \delta(x),$$

which is a contradiction. This completes the proof of the result. □

In light of this lemma, we define, for each triplet of users $u_a, u_b, u_c \in U$, and any placement of facility $x \in \mathbb{R}^2$ by P1, the indicator variable,

$$T_{abc}(x) = \begin{cases} 1 & \text{if } C_a(x)\bigcap C_b(x)\bigcap C_c(x) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

Let $\boldsymbol{T}(x) = ((T_{abc}(x)))_{1\leq a,b,c\leq |U|}$ be the 3-dimensional array with cardinality $|U| \times |U| \times |U|$, where each cell $T_{abc}(x)$ is defined as above. Then by Lemma 4.1, the following observation is immediate.

Observation 4.1. *If the points x, y belong to the same cell of $\mathcal{T}(F, S)$ and the two arrays $\boldsymbol{T}(x) = \boldsymbol{T}(y)$ in every coordinate, then $\delta(x) = \delta(y)$.*

As in Section 3, our goal is to tessellate $\mathcal{T}(F, S)$ in to finer set of cells such that for any two points x and y in the same cell, $\boldsymbol{T}(x) = \boldsymbol{T}(y)$. As in (3.1), define, for $u_a, u_b, u_c \in U$,

$$\mathcal{T}(u_a, u_b, u_c) = \left\{x \in \mathbb{R}^2 : C_a(x)\bigcap C_b(x)\bigcap C_c(x) = \varnothing\right\}. \tag{4.1}$$

Definition 4.1. Given any placement x by P1 and a user $u_a \in U_{FS}(x)\bigcup U_{F\setminus x}$, the disc $C(x)$ is called an *old disc* if it is centered at u_a and passes through some facility $f_j \in F$, where f_j is the facility closest to u_a among the set of facilities $F\bigcup\{x\}$, that is, $u_a \in U_{F\setminus x}$ (recall (2.2)). The disc $C(x)$ is called a *new disc* if it is centered at u_a and passes through x, that is $u \in U_{FS}(x)$ (recall (2.1)).

Let $u_a, u_b, u_c \in U_{FS}(\lambda)\bigcup U_{F\setminus\lambda}$, for some cell λ of $\mathcal{T}(F, S)$. For $S \subseteq \{a, b, c\}$, define the following sets:

$$\mathcal{T}_S(u_a, u_b, u_c) = \{x \in \mathcal{T}(u_a, u_b, u_c) : C_s(x) \text{ is new, if } s \in S, \text{ and } C_s(x) \text{ is old, if } s \notin S\},$$

where new/old are as defined in Definition 4.1. Note that

$$\mathcal{T}(u_a, u_b, u_c) = \bigcup_{S\subseteq\{a,b,c\}} \mathcal{T}_S(u_a, u_b, u_c). \tag{4.2}$$

Therefore, to compute the set $\mathcal{T}(u_a, u_b, u_c)$, it suffices to understand the sets $\mathcal{T}_S(u_a, u_b, u_c)$, for $S \subseteq \{a, b, c\}$. To this end, we have the following lemma:

Lemma 4.2. *Let $u_a, u_b, u_c \in U_{FS}(\lambda)\bigcup U_{F\setminus\lambda}$, for some cell λ of $\mathcal{T}(F, S)$. Then*

(a) $\mathcal{T}_\varnothing(u_a, u_b, u_c) = (C_a\bigcup C_b\bigcup C_c)^c$.

(b) $\mathcal{T}_{\{a,b,c\}}(u_a, u_b, u_c) = \Delta(u_a, u_b, u_c)\bigcap(C_a\bigcap C_b\bigcap C_c)$, where $\Delta(u_a, u_b, u_c)$ is the triangle formed by $u_a, u_b,$ and u_c.

(c) $\mathcal{T}_{\{c\}}(u_a, u_b, u_c) = D_{\{c\}}(a,b) \bigcap C_c \backslash (C_a \bigcup C_b)$*, where* $D_{\{c\}}(a,b)$ *is the closed disc centered at* u_c *and passing through the point in* $C_a \bigcap C_b$ *closest to* u_c *in the* L_2 *metric. The sets* $\mathcal{T}_{\{a\}}(u_a, u_b, u_c)$*, and* $\mathcal{T}_{\{b\}}(u_a, u_b, u_c)$ *are defined similarly.*

Proof. The sets $\mathcal{T}_{\varnothing}(u_a, u_b, u_c)$ and $\mathcal{T}_{\{a,b,c\}}(u_a, u_b, u_c)$ can be derived easily from the definitions.

To characterize $\mathcal{T}_{\{c\}}(u_a, u_b, u_c)$, let p_0 be the point in $C_a \bigcap C_b$ which is closest to u_c in the L_2 metric. Note that $D_{\{c\}}(a,b)$ is the closed disk centered at u_c and passing through p_0. Note that if $x \in D_{\{c\}}(a,b) \cap C_c \backslash (C_a \bigcup C_b)$, then $C_a(x) \bigcap C_b(x) \bigcap C_c(x) = \varnothing$, and if $x \notin \left(D_{\{c\}}(a,b)\right)^c \cap C_c \backslash (C_a \bigcup C_b)$, then $p_0 \in C_a(x) \bigcap C_b(x) \bigcap C_c(x)$. This implies, $\mathcal{T}_{\{c\}}(u_a, u_b, u_c) = D_{\{c\}}(a,b) \bigcap C_c \backslash (C_a \bigcup C_b)$. □

The above result shows that the sets $\mathcal{T}_S(u_a, u_b, u_c)$, for $|S| \neq 2$, can be easily computed in $O(1)$ time. Thus, it remains to compute $\mathcal{T}_S(u_a, u_b, u_c)$, for $|S| = 2$. In this case, the sets are more complicated, and it is difficult to explicitly describe the structure of these sets geometrically. The following lemma shows that these sets can also be computed in $O(1)$ time, and through the proof of this lemma the precise geometry of these sets can be described.

Lemma 4.3. *Let* $u_a, u_b, u_c \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda}$*, for some cell* λ *of* $\mathcal{T}(F, S)$*. Then the boundary of* $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ *is made up of* $O(1)$ *circular arcs, and can be computed in* $O(1)$ *time.*

4.1 Proof of Lemma 4.3:

For a set $A \subseteq \mathbb{R}^2$ denote by $\overline{A}$ and ∂A, the closure and the boundary of A, respectively. For example, $\overline{C_a(x)}$ is the closed disc centered at u_a and passing through the facility closest to u_a among the facilities in $F \bigcup S \bigcup \{x\}$. We begin with the following simple observation:

Observation 4.2. *Let* $u_a, u_b, u_c \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda}$*, for some cell* λ *of* $\mathcal{T}(F, S)$*. Then*

$$\partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) = \left\{p \in (C_a \bigcap C_b) \backslash C_c : \left|\overline{C_a(p)} \bigcap \overline{C_b(p)} \bigcap \overline{C_c(p)}\right| = 1\right\}.$$

Moreover, the set $\partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ *is symmetric about the line joining* u_a, u_b*, that is, if* $p \in \partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ *then its reflection about the line joining* u_a, u_b $p^\perp$ *also belongs to* $\partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$*.*

Proof. Suppose that $p \in (C_a \bigcap C_b) \backslash C_c$ is such that $\overline{C_a(p)} \bigcap \overline{C_b(p)} \bigcap \overline{C_c(p)} = \{q\}$, where q is a point on the boundary of $\overline{C_a(p)} \bigcap \overline{C_b(p)}$. Next, note that for any point x in the interior of $\overline{C_a(p)} \bigcap \overline{C_b(p)}$, $\overline{C_a(x)} \bigcap \overline{C_b(x)}$ is a proper subset of $\overline{C_a(p)} \bigcap \overline{C_b(p)}$, and $C_a(x) \bigcap C_b(x) \bigcap C_c(x) = \varnothing$, since $C_c(x) = C_c$ is an old circle. Therefore, every point in the interior of $\overline{C_a(p)} \bigcap \overline{C_b(p)}$ belongs to $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$. Similarly, if x lies outside $\overline{C_a(p)} \bigcup \overline{C_b(p)}$, then q is in the interior of $\overline{C_a(x)} \bigcap \overline{C_b(x)}$, and $C_a(x) \bigcap C_b(x) \bigcap C_c(x) \neq \varnothing$. Therefore, every open ball centered at p intersects both $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ and the complement $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)^c$, that is, $p \in \partial\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$.

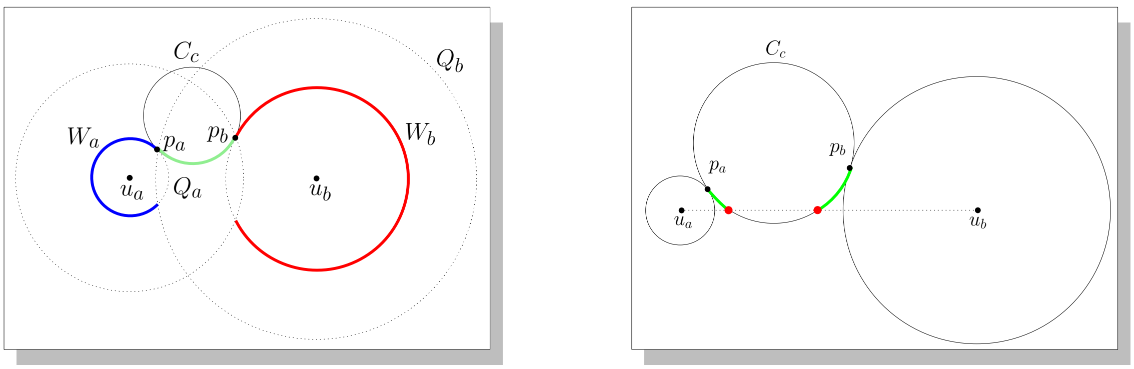
Similarly, it can be shown that if $\overline{C_a(p)} \bigcap \overline{C_b(p)} \bigcap \overline{C_c(p)}$ contains zero points or more than 1 point, then the point p does not belong to the boundary of $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$.

Finally, to prove the symmetry, observe that for any point $p \in (C_a \bigcup C_b) \backslash C_c$, the boundaries of the discs $C_a(p)$ and $C_b(p)$ intersect at points p and $p^\perp$. Hence, if for any point p, $\overline{C_a(p)} \bigcap \overline{C_b(p)} \bigcap \overline{C_c(p)}$ is a singleton set then $\overline{C_a(p^\perp)} \bigcap \overline{C_b(p^\perp)} \bigcap \overline{C_c(p^\perp)}$ is also singleton, since $C_c(p) = C_c(p^\perp) = C_c$ is an old circle. This completes the proof. □

Note that for $p \in \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$, the circle $C_c(p)$ is an old circle, which does not depend on p. Hereafter, we will drop the dependence on p and denote this circle by C_c. Now, define sets

$$\begin{aligned} W_a &:= \left\{ p \in \partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) : \left| \overline{C_b(p)} \bigcap \overline{C_c} \right| = 1 \right\}, \\ W_b &:= \left\{ p \in \partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) : \left| \overline{C_a(p)} \bigcap \overline{C_c} \right| = 1 \right\}, \\ W_c &:= \left\{ p \in \partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) : \left| \overline{C_b(p)} \bigcap \overline{C_b(p)} \right| = 1 \right\}. \end{aligned} \tag{4.3}$$

The sets W_a, W_b, W_c can obtained using the above observation:



(a) (b)

Figure 5: (a) The sets W_a and W_b, and (b) the set W_c, as defined in (4.3).

W_a/W_b**:** Let the point closest to u_a in $\overline{C_c}$ be p_a, and let Q_a and Q_b be the circles with centers at u_a and u_b and radii $d_2(u_a, p_a)$) and $d_2(u_a, p_b)$), respectively. Then for all points $p \in Q_a \backslash Q_b$ $\overline{C_a(p)} \bigcap \overline{C_b(p)} \bigcap \overline{C_c} = \{p_a\}$, that is, $W_a = (Q_a \backslash Q_b) \bigcap (C_a \bigcap C_b) \backslash C_c$ (blue curve in Figure 5(a)). The set W_b can be obtained similarly (red curve in Figure 5(a)).

W_c**:** In this case, it is easy to see that $W_c = [u_a, u_b] \bigcap \partial C_c$. These are the two red points in Figure 5(b). This set is empty in Figure 5(a).

Finally, let

$$W_0 := \partial \mathcal{T}_{\{a,b\}}(u_a, u_b, u_c) \backslash \left(W_a \bigcup W_b \bigcup W_c \right). \tag{4.4}$$

This is the set of all points p such that $\overline{C_a(p)} \bigcap \overline{C_b(p)} \bigcap \overline{C_c}$ is a singleton set, but the intersection of no two of them is a singleton.

W_0**:** Recall that p_a is the point closest to u_a in $\overline{C_c}$. Let p_b be the point closest to u_b in $\overline{C_c}$. If the line joining u_a, u_b does intersect C_c, then W_0 is the arc between p_a and p_b (arc colored in green in Figure 5(a)) and its reflection on the line joining u_a and u_b (see Observation 4.2), intersected with $(C_a \bigcap C_b) \backslash C_c$. Otherwise, u_a, u_b does intersect C_c, and the segment of the arc between p_a and p_b above the line joining u_a, u_b (green arcs in Figure 5(b)) and its reflection on the line joining u_a and u_b gives the set W_0, when intersected with $(C_a \bigcap C_b) \backslash C_c$.

The sets W_a, W_b, W_c, and W_0 together make up the boundary of the set $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$. Putting these together, we get the shapes in Figure 6, depending on whether or not the line joining u_a and u_b intersects the disc C_c. The set $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ can be obtained when the shapes (the regions bounded by the blue curves) in Figure 6 are intersected with $C_a \bigcap C_b \backslash C_c$. This shows that the boundary of $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ is made up of $O(1)$ circular-arcs. Therefore, the set $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ can be computed in $O(1)$ time, which completes the proof of Lemma 4.3.

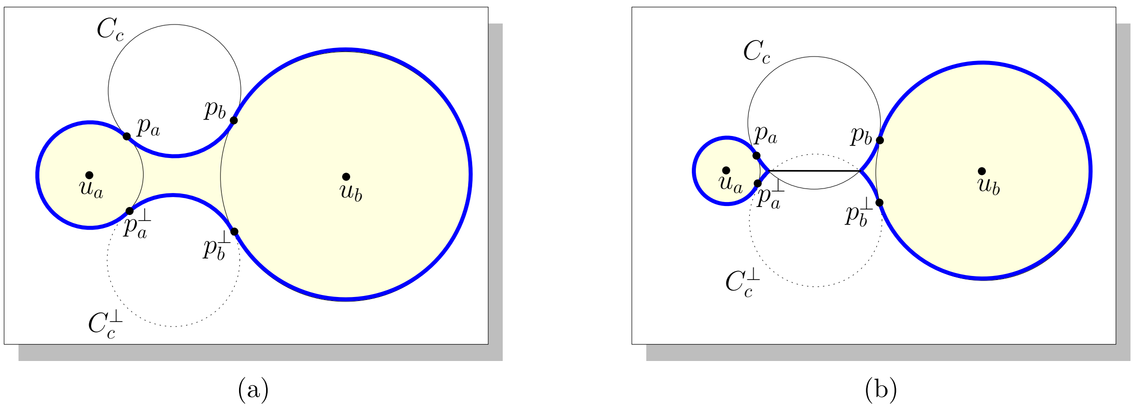


Figure 6: The set $\mathcal{T}_{\{a,b\}}(u_a, u_b, u_c)$ (a) when the line joining u_a and u_b does not intersect C_c, and (b) when line joining u_a and u_b intersects C_c.

4.2 Completing the Proof of Theorem 1.2: Consider the arrangement in $\mathbb{R}^2$ generated by the collection of sets $\{\mathcal{T}(u_a, u_b, u_c) : u_a, u_b, u_c \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda} \text{ and } \lambda \in \mathcal{T}(F, S)\}$. Lemmas 4.2 and 4.3 show that the arrangement is made up of $O(n^3)$ discs, circular arcs, and line segments, which can be computed in $O(n^3)$ time. These intersect to generate $O(n^6)$ cells in $\mathcal{T}(F, S)$, and by Observation 4.1, all points in a particular cell have the same effective depth. Note that effective depth of a cell can be computed in $O(n^2)$ time, by finding the maximum depth of the arrangement $\{C_a(x) : u_a \in U_{FS}(\lambda) \bigcup U_{F\backslash\lambda}\}$, for any point x in the cell [9, Theorem 1]. Therefore, the optimal strategy of P1 in the L_2 metric can be computed in $O(n^8)$ time. The corresponding level sets can be computed in the same running time.

Acknowledgement: The authors thank the anonymous referees for their helpful comments, which improved the presentation of the paper. The authors also thank Frank Plastria for bringing to their attention the reference [24].

References

[1] Hee-Kap Ahn, Siu-Wing Cheng, Otfried Cheong, Mordecai J. Golin, and René van Oostrum. Competitive facility location: the Voronoi game. *Theor. Comput. Sci.*, 310(1-3):457–467, 2004.

[2] Boris Aronov and Sariel Har-Peled. On approximating the depth and related problems. *SIAM J. Comput.*, 38(3):899–921, 2008.

[3] Sayan Bandyapadhyay, Aritra Banik, Sandip Das, and Hirak Sarkar. Voronoi game on graphs. *Theor. Comput. Sci.*, 562:270–282, 2015.

[4] Aritra Banik, Bhaswar B. Bhattacharya, and Sandip Das. Optimal strategies for the one-round discrete Voronoi game on a line. *Journal of Combinatorial Optimization*, 2012.

[5] Aritra Banik, Jean-Lou De Carufel, Anil Maheshwari, and Michiel H. M. Smid. Discrete voronoi games and ϵ-nets, in two and three dimensions. *Comput. Geom.*, 55:41–58, 2016.

[6] Aritra Banik, Sandip Das, Anil Maheshwari, and Michiel H. M. Smid. The discrete voronoi game in a simple polygon. In Ding-Zhu Du and Guochuan Zhang, editors, *Computing and Combinatorics, 19th International Conference, COCOON 2013, Hangzhou, China, June 21-23, 2013. Proceedings*, volume 7936 of *Lecture Notes in Computer Science*, pages 197–207. Springer, 2013.

[7] Bhaswar B. Bhattacharya. Maximizing Voronoi regions of a set of points enclosed in a circle with applications to facility location. *J. Math. Model. Algorithms*, 9(4):375–392, 2010.

[8] Bhaswar B. Bhattacharya and Subhas C. Nandy. New variations of the maximum coverage facility location problem. *European Journal of Operational Research*, 224(3):477–485, 2013.

[9] Sergio Cabello, José Miguel Díaz-Báñez, Stefan Langerman, Carlos Seara, and Inmaculada Ventura. Facility location problems in the plane based on reverse nearest neighbor queries. *European Journal of Operational Research*, 202(1):99–106, 2010.

[10] Otfried Cheong, Alon Efrat, and Sariel Har-Peled. Finding a guard that sees most and a shop that sells most. *Discrete & Computational Geometry*, 37(4):545–563, 2007.

[11] Otfried Cheong, Sariel Har-Peled, Nathan Linial, and Jirí Matousek. The one-round Voronoi game. *Discrete & Computational Geometry*, 31(1):125–138, 2004.

[12] Frank K. H. A. Dehne, Rolf Klein, and Raimund Seidel. Maximizing a Voronoi region: the convex case. *Int. J. Comput. Geometry Appl.*, 15(5):463–476, 2005.

[13] Tammy Drezner. A review of competitive facility location in the plane. *Logistics Research*, 7(1):114:1–114:12, 2014.

[14] Zvi Drezner and Horst W. Hamacher. *Facility location - applications and theory.* Springer, 2002.

[15] Christoph Dürr and Nguyen Kim Thang. Nash equilibria in Voronoi games on graphs. In Lars Arge, Michael Hoffmann, and Emo Welzl, editors, *ESA*, volume 4698 of *Lecture Notes in Computer Science*, pages 17–28. Springer, 2007.

[16] Horst A. Eiselt and Gilbert Laporte. Competitive spatial models. *European Journal of Operational Research*, 39(3):231–242, 1989.

[17] Horst A. Eiselt, Gilbert Laporte, and Jacques F. Thisse. Competitive location models: A framework and bibliography. *Transportation Science*, 27:44–54, 1993.

[18] Sándor P. Fekete and Henk Meijer. The one-round Voronoi game replayed. *Comput. Geom.*, 30(2):81–94, 2005.

[19] Dániel Gerbner, Viola Mészáros, Dömötör Pálvölgyi, Alexey Pokrovskiy, and Günter Rote. Advantage in the discrete Voronoi game. *arXiv preprint arXiv:1303.0523*, 2013.

[20] Harold Hotelling. Stability in competition. *The Economic Journal*, 39(153):41–57, 1929.

[21] Stefan Langerman and William L. Steiger. Optimization in arrangements. In Helmut Alt and Michel Habib, editors, *STACS*, volume 2607 of *Lecture Notes in Computer Science*, pages 50–61. Springer, 2003.

[22] J. Matoušek. Computing the center of planar point sets. *Computational Geometry: Papers from the DIMACS Special Year*, pages 221–230, 1991.

[23] Igor Pak. *Lectures on Discrete and Polyhedral Geometry.* 2010.

[24] F. Plastria. Avoiding cannibalization and/or competitor reaction in planar single facility location. *Journal of the Operations Research Society of Japan*, 48 (2):148–157, 2005.

[25] Sachio Teramoto, Erik D. Demaine, and Ryuhei Uehara. The Voronoi game on graphs and its complexity. *J. Graph Algorithms Appl.*, 15(4):485–501, 2011.