

Homework 2, Stat 541: Linear Algebra

Due Fri, Sept 25, 2009, 12 Noon

Your Name: (replace this with your name)

September 22, 2009

Instructions: Edit this LaTeX file by inserting your solutions after each problem statement. Generate a PDF file from it and e-mail the PDF to the usual class gmail address. You should not just answer the questions but also give evidence or even proof. You can discuss the homework with each other in general terms, but not with previous years' students of Stat 541. Also, you must write your own solutions and not copy from anyone.

1. Interpretations of matrix multiplication: Assume \mathbf{X} is of size $n \times p$ and \mathbf{B} of size $p \times q$.

(a) If $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p)$, $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p)^T$, express

$$\mathbf{XB} = \sum_{j=1, \dots, p} \mathbf{x}_j \mathbf{b}_j^T$$

A:

(b) If $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)^T$, $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_q)$, express

$$(\mathbf{XB})_{i,k} = \mathbf{x}_i^T \mathbf{b}_k$$

A:

2. Write the Euclidean inner product as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ and the Euclidean squared norm as $\|\mathbf{x}\|^2 = \mathbf{x}^T \mathbf{x}$. What is the interpretation of

$$\mathbf{y} \mapsto \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2} \mathbf{x} ?$$

A:

3. If \mathbf{A} is a $n \times n$ matrix, when does it satisfy $\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{Ay} \rangle$ for all \mathbf{x} and all \mathbf{y} ?

A:

4. If $\|\mathbf{x}\|^2 = 1$, what is the interpretation of $\mathbf{P} = \mathbf{xx}^T$? In other words, what does $\mathbf{y} \mapsto \mathbf{Py}$ do? (Assume the vectors are compatible: $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.)

A:

5. Same question if $\|\mathbf{x}\|^2 \neq 1$?

A:

6. Assuming $\|\mathbf{x}\|^2 = 1$, verify that $\mathbf{P} = \mathbf{xx}^T$ satisfies (1) $\mathbf{PP} = \mathbf{P}$ and (2) $\mathbf{P}^T = \mathbf{P}$. Property (1) is called idempotence, (2) is called symmetry.

A: (1)

(2)

7. Assuming $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 = 1$, under what condition is $\mathbf{P} = \mathbf{xx}^T + \mathbf{yy}^T$ also idempotent? Is it always symmetric?

A:

8. If \mathbf{P} ($n \times n$) is idempotent and symmetric, does the same hold for $\mathbf{Q} = \mathbf{I} - \mathbf{P}$?

A:

9. If \mathbf{P} is idempotent, when is it non-singular?

A:

10. Can you give idempotence an intuitive meaning? It may help to look at the range space and null space of an idempotent map \mathbf{P} .

A:

11. Assuming \mathbf{P} and \mathbf{A} are of the same size $n \times n$ and \mathbf{A} is non-singular, what is the meaning of $\mathbf{Q} = \mathbf{APA}^{-1}$? (Hint: coordinate transformation.)

A:

12. If \mathbf{P} is idempotent, when is $\mathbf{Q} = \mathbf{APA}^{-1}$, too?

A:

13. If \mathbf{P} is symmetric, when is $\mathbf{Q} = \mathbf{A}\mathbf{P}\mathbf{A}^{-1}$, too?

A:

14. If \mathbf{P} is idempotent, and if its range space and null space are orthogonal to each other, show that P is symmetric. Show, conversely, that if P is symmetric these spaces are orthogonal.

A:

15. If \mathbf{X} is of size $n \times p$ ($n \geq p$) and of rank p , let $\mathbf{P} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$. Is \mathbf{P} idempotent and symmetric?

A:

16. What are the range space and the null space of \mathbf{P} in the previous problem?

A:

17. If \mathbf{X} is of size $n \times p$ ($n \geq p$) and full rank, exactly when is $\mathbf{P} = \mathbf{X}\mathbf{X}^T$ idempotent? When is this matrix symmetric?

A:

18. If \mathbf{P} is idempotent, when does the inverse of $\mathbf{I} + c\mathbf{P}$ exist? What is it?

A:

19. Subject the identity for the inverse of $\mathbf{I} + c\mathbf{P}$ obtained in the previous item to a coordinate transformation with a matrix \mathbf{A} (as earlier). What general identity do you obtain?

A: