

Homework 2, Stat 541: Linear Algebra, Due Fri, Sept 21

Student Name: (replace this with your name)

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Instructions: Edit this LaTeX file with your solutions and generate a PDF file from it. E-mail the PDF to the usual class gmail address.

1. Interpretations of matrix multiplication: Assume \mathbf{X} is of size $n \times p$ and \mathbf{B} of size $p \times q$.

(a) If $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p)$, $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p)^T$, express

$$\mathbf{XB} = \sum_{j=1, \dots, p} \dots$$

(b) If $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)^T$, $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_q)$, express

$$(\mathbf{XB})_{i,k} = \dots$$

2. Write the Euclidean inner product as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ and the Euclidean squared norm as $\|\mathbf{x}\|^2 = \mathbf{x}^T \mathbf{x}$. What is the interpretation of

$$\mathbf{y} \mapsto \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2} \mathbf{x} ?$$

3. If \mathbf{A} is a $n \times n$ matrix, when does it satisfy $\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{AY} \rangle$ for all \mathbf{x} and all \mathbf{y} ?

4. If $\|\mathbf{x}\|^2 = 1$, what is the interpretation of $\mathbf{P} = \mathbf{xx}^T$? In other words, what does $\mathbf{y} \mapsto \mathbf{Py}$ do? (Assume the vectors are compatible: $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.)

5. Same question if $\|\mathbf{x}\|^2 \neq 1$?

6. Assuming $\|\mathbf{x}\|^2 = 1$, verify that $\mathbf{P} = \mathbf{xx}^T$ satisfies (1) $\mathbf{PP} = \mathbf{P}$ and (2) $\mathbf{P}^T = \mathbf{P}$. Property (1) is called idempotence, (2) is called symmetry.

7. Assuming $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 = 1$, under what condition is $\mathbf{P} = \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T$ also idempotent? Is it always symmetric?
8. If \mathbf{P} ($n \times n$) is idempotent and symmetric, does the same hold for $\mathbf{Q} = \mathbf{I} - \mathbf{P}$?
9. If \mathbf{P} is idempotent, when is it non-singular?
10. Can you give idempotence an intuitive meaning? It may help to look at the range space and null space of an idempotent map \mathbf{P} .
11. Assuming \mathbf{P} and \mathbf{A} are of the same size $n \times n$ and \mathbf{A} is non-singular, what is the meaning of $\mathbf{Q} = \mathbf{A}\mathbf{P}\mathbf{A}^{-1}$? (Hint: coordinate transformation.)
12. If \mathbf{P} is idempotent, when is $\mathbf{Q} = \mathbf{A}\mathbf{P}\mathbf{A}^{-1}$, too?
13. If \mathbf{P} is symmetric, when is $\mathbf{Q} = \mathbf{A}\mathbf{P}\mathbf{A}^{-1}$, too?
14. If \mathbf{P} is idempotent, and if its range space and null space are orthogonal to each other, show that \mathbf{P} is symmetric. Show that the reverse is true, too.
15. If \mathbf{X} is of size $n \times p$ ($n \geq p$) and of rank p , let $\mathbf{P} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$. Is \mathbf{P} idempotent and symmetric?
16. What are the range space and the null space of \mathbf{P} ?
17. If \mathbf{X} is of size $n \times p$ ($n \geq p$), when is $\mathbf{P} = \mathbf{X}\mathbf{X}^T$ idempotent? When is it symmetric?
18. If \mathbf{P} is idempotent, when does the inverse of $\mathbf{I} + c\mathbf{P}$ exist? What is it?
19. Subject the identity for the inverse of $\mathbf{I} + c\mathbf{P}$ obtained in the previous item to a coordinate transformation with a matrix \mathbf{A} (as earlier). What general identity do you obtain?