STAT 430/510: Lecture 17

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### Updates

- HW4 is due **today**.
- HW2 grades are (finally) up, as are the HW3 solutions. I’ll get around to grading that soon.
- Discuss HW5.
Formalization

- **Def:** The *covariance* between two r.v.’s $X$ and $Y$ is defined as

\[
\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] 
\]

- An alternative form of *covariance* is

\[
\text{Cov}(X, Y) = E[XY] - E[X]E[Y] 
\]

- Remembering back, $\text{Cov}$ comes up when we look at $X$ and $Y$ not independent and . . .

\[
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) 
\]
Properties

- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$.
- $\text{Cov}(X, X) = \text{Var}(X)$.
- $\text{Cov}(aX, bY) = ab \cdot \text{Cov}(X, Y)$.
- $\text{Cov}(\sum_i X_i, \sum_j Y_j) = \sum_i \sum_j \text{Cov}(X_i, Y_j)$.
- $\text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j)$.
- If $X_i$ are pairwise independent, then

$$\text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i)$$
Example 1

Let

\[ X = \begin{cases} 
1 & \text{w.p. } 1/3 \\
0 & \text{w.p. } 1/3 \\
-1 & \text{w.p. } 1/3 
\end{cases} \]

and

\[ Y = \begin{cases} 
1 & \text{if } X = 0 \\
0 & \text{if } X \neq 0 
\end{cases} \]

**Question**: What is $XY$ and $E[XY]$?

**Solution**: $XY = 0$, because $Y$ is 0 if $X$ is not and $Y$ is not 0 if $X$ is.

Thus, the $E[XY] = 0$.

**Question**: Are $X$ and $Y$ independent?

**Solution**: No, even though $Cov(X, Y) = 0$. 

Example 2

Think back to example 2c, in this chapter.

Let $X_1, \ldots, X_n$ be i.i.d r.v.’s having expected values $\mu$ and variance $\sigma^2$.

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ be the sample mean.

Let the quantities $X_i - \bar{X}$ be called deviations.

These are all the differences between the individual data and the sample mean.

The r.v. $S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}$ is called the sample variance.

**Question:** What is $\text{Var}(\bar{X})$?
Example 2 (cont.)

Solution:

\[
\text{Var}(\bar{X}) = \text{Var} \left( \sum_{i=1}^{n} \frac{X_i}{n} \right) \\
= \left( \frac{1}{n} \right)^2 \text{Var} \left( \sum_{i=1}^{n} X_i \right) \\
= \left( \frac{1}{n} \right)^2 \sum_{i=1}^{n} \text{Var}(X_i) \text{ by independence} \\
= \frac{\sigma^2}{n}
\]

Question: \( E[S^2] \)?
Example 2 (cont.)

- **Solution**: Start off by multiplying by \((n - 1)\) to eliminate the constant. Then . . .

\[
\begin{align*}
(n - 1)S^2 &= \sum_{i=1}^{n} (X_i - \bar{X})^2 \\
&= \sum_{i=1}^{n} (X_i - \mu + \mu - \bar{X})^2 \text{ (adding 0)} \\
&= \sum_{i=1}^{n} (X_i - \mu)^2 + \sum_{i=1}^{n} (\bar{X} - \mu)^2 - 2(\bar{X} - \mu) \sum_{i=1}^{n} (X_i - \mu) \\
&= \sum_{i=1}^{n} (X_i - \mu)^2 + n(\bar{X} - \mu)^2 - 2(\bar{X} - \mu)n(\bar{X} - \mu) \\
&= \sum_{i=1}^{n} (X_i - \mu)^2 - n(\bar{X} - \mu)^2
\end{align*}
\]
Example 2 (cont.)

- Now, take the expectation of that:

\[(n - 1)E[S^2] = \sum_{i=1}^{n} E[(X_i - \mu)^2] - nE[(\bar{X} - \mu)^2] = n\sigma^2 - n\text{Var}(\bar{X}) = (n - 1)\sigma^2\]

- Thus, \(E[S^2] = \sigma^2\), which is what we would want when estimating the variance of some data.
Example 3

Let \( X = X_1 + \ldots + X_n \), where \( X_1, \ldots, X_n \) are i.i.d Bernoulli trials with prob. of success \( p \). Then, \( X \) is . . .

Binomial with parameters \((n, p)\). We’ve talked about the variance of a Binomial r.v. before; this is how we can prove it:

\[
\text{Var}(X) = \text{Var}\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i)
\]
Example 3 (cont.)

Note that:

$$Var(X_i) = E[X_i^2] - (E[X_i])^2$$

$$= E[X_i] - (E[X_i])^2$$

$$= p - p^2$$

Then,

$$Var(X) = np(1 - p)$$
Formalization

- **Def:** The correlation of two r.v.’s $X$ and $Y$, denoted by $\rho(X, Y)$, is defined as

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}$$

- Note that

$$-1 \leq \rho(X, Y) \leq 1$$

- If $\rho(X, Y) = 0$, then $X$ and $Y$ are said to be **uncorrelated**.

- $X$ and $Y$ are uncorrelated if and only if

$$E[XY] = E[X]E[Y]$$

- Correlation indicates the strength of a linear relationship between two variables.
Properties

- Covariance depends on the unit of measurement, thus, making it difficult to interpret a computed value.
- Correlation is scale independent:
  - $\rho$ is not affected by a linear change in the units of measurement (e.g. pound $\leftarrow$ kilo).
  - If $b$ and $d$ are both positive or both negative, then $\rho(a + bX, c + dY) = \rho(X, Y)$.
- When $|\rho(X, Y)| = 1$, then $Y = a + bX$ for some $a, b$.
- If $X$ and $Y$ are independent, then $\rho = 0$.
- However, $\rho = 0$ does not imply independence between $X$ and $Y$, as was seen in the previous example.
Example 4

Let

$$I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}, \quad I_B = \begin{cases} 1 & \text{if } B \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

**Question**: What is $Cov(I_A, I_B)$?
Example 4 (cont.)

**Solution:** We know that:

\[
E[I_A] = P(A) \\
E[I_B] = P(B) \\
E[I_A I_B] = P(AB)
\]

Then,

\[
\text{Cov}(I_A, I_B) = P(AB) - P(A)P(B) \\
= P(B)[P(A|B) - P(A)]
\]

The indicator variables for \( A \) and \( B \) are either positively correlated, uncorrelated, or negatively correlated, depending on how \( B \) affects the probability of \( A \) occurring.
Example 5

- Let $X$ be the number of 1’s and $Y$ the number of 2’s that occur in $n$ rolls of a fair die.

- **Question:** What is $\text{Corr}(X, Y)$?

- **Solution:** $X$ and $Y$ and be talked about as the sum of . . .

$$X_i = \begin{cases} 1 & \text{roll } i \text{ lands on 1} \\ 0 & \text{otherwise} \end{cases}, \quad Y_i = \begin{cases} 1 & \text{roll } i \text{ lands on 2} \\ 0 & \text{otherwise} \end{cases}$$
Example 5 (cont.)

Then, the covariance between $X_i$ and $Y_i$ is . . .

$$\text{Cov}(X_i, Y_i) = E[X_i Y_j] - E[X_i]E[Y_j]$$

$$= \begin{cases} 
-\frac{1}{36} & i = j \\
\frac{1}{36} - \frac{1}{36} = 0 & i \neq j 
\end{cases}$$

So, the covariance must be

$$\text{Cov}(\sum_i X_i, \sum_j Y_j) = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$

$$= -\frac{n}{36}$$
Example 5 (cont.)

- To finish this off, we need the variance of $X$ and $Y$:

$$\text{Var}(X) = \text{Var}\left(\sum_{i} X_i\right)$$

$$= \sum_{i} \text{Var}(X_i)$$

$$= \frac{15}{66} = \frac{5}{36}$$

$$= \text{Var}(Y)$$

- So, the correlation must be:

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$= \frac{-n/36}{(5n)/36} = -\frac{1}{5}$$
Formalization

- **Def**: If $X$ and $Y$ are jointly discrete r.v.’s, then the **conditional expectation** of $X$ given $Y = y$, for all values of $y$ s.t. $p_Y(y) > 0$, is defined as

$$ E[X|Y = y] = \sum_x x \cdot p_{X|Y}(x|y) $$

- **Def**: If $X$ and $Y$ are jointly continuous r.v.’s, then the **conditional expectation** of $X$ given $Y = y$, provided that $f_Y(y) > 0$, is defined as

$$ E[X|Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y) \, dx $$
Properties

- **Proposition 7.5.1:**

\[ E[X] = E[E[X|Y]] \]

That is, for a discrete r.v.,

\[ E[X] = \sum_{y} E[X|Y = y]P(Y = y) \]

And, for a continuous r.v.,

\[ E[X] = \int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)dy \]

- **Law of Total Variance:**

\[ Var(X) = Var(E(X|Y)) + E(Var(X|Y)) \]
Example 6

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours. The third door leads to a tunnel that will return him to the mine after 7 hours.

**Question:** If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?
Example 6 (cont.)

**Solution:** Let $X$ denote the amount of time (in hours) until the miner reaches safety and let $Y$ denote the door he initially chooses. So, $E[X]$ is . . .

\[
\]

\[
= \frac{1}{3}(E[X|Y=1] + E[X|Y=2] + E[X|Y=3])
\]

What are each of those conditional expectations?

\[
E[X|Y=1] = 3
\]
\[
E[X|Y=2] = 5 + E[X]
\]
\[
E[X|Y=3] = 7 + E[X]
\]
Thus, plugging this back in, we get . . .

\[ E[X] = \frac{1}{3}(3 + 5 + E[X] + 7 + E[X]) \]

\[ \Rightarrow \frac{1}{3}E[X] = 5 \]

or \[ E[X] = 15 \]
Now, covered section 7.4 and are on 7.5.

Putting up HW5 after class.