Homework 1 is now up and is due Wed. June 2nd.

Be sure to include all of your work and thought process when turning in the homework to ensure full credit.

Everyday after class, I’ll make sure to leave time for any questions you have, either on the lectures, the book, or homework, but feel free to ask your questions during class as others may have the same question.

The midterm is likely to be on June 15th. Any conflicts?

We now have a site up on webcafe.
  - Material, assignments, etc. will still be on course page.
  - Grades, however, will be done in webcafe.
Example 1

- Some of the confusion yesterday lied in the difference between the order amongst "elements" and "groups." Here is an example illustrating this exact point.

- **Question:** Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and B in another. How many different divisions are possible?
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- **Solution**:

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\binom{10}{5, 5}
\]
Example 1 (cont.)

**Question**: Let say those 10 children divide themselves into two teams of 5 each. How many different divisions are possible?
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- **Solution**: These are two different questions, with the difference being that there is no A and B team, just the division of two groups.

That is, if person a, b, c, d, and e were in one group and person f, g, h, i, and j in another, then there would be no way to tell an ordering of the two unless we made one group have the label A and the other have the label B.
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  Thus, the answer is:

  \[
  \frac{\binom{10}{5,5}}{P_{2,2}} = \frac{10!/5!5!}{2!} = 126
  \]
Example 2

- One more example that uses this concept.
- **Question**: How many different ways can you make a 5-card hand have two pair with a 52-card deck?
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**Step 1:** Choose the value of the "first" pair: \( \binom{13}{1} \).

**Step 2:** Choose two of the four cards with that value: \( \binom{4}{2} \).
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  - **Step 3**: Repeat **Step 1** and 2 with one value missing for the "second" pair: \( \binom{12}{1} \binom{4}{2} \).
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  - **Step 4**: With one card left, the requirement is that it (a) can’t be a card already chosen and (b) can’t have the value of either of the two pairs: \( \binom{52-4-4-4}{1} \).
Example 2 (cont.)

Why are we not done at this point?

Because the "first" pair and "second" pair don't have labels and are indistinguishable from each other.

Step 5: Divide by the number of ordered arrangements the pairs can take: \(P^2_2, 2^2\).

Step 6: Put it all together to get \((\binom{13}{1})\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{44}{1}\cdot 1^2!\).

The other way of getting to the same answer is putting it like this: \((\binom{13}{2})\binom{4}{2}\binom{4}{2}\binom{44}{1}\).

This eliminates the ordering involved in picking the pairs.
Example 2 (cont.)

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$$\left(\frac{13}{1}\right)\left(\frac{4}{2}\right)\left(\frac{12}{1}\right)\left(\frac{4}{2}\right)\left(\frac{44}{1}\right) \cdot \frac{1}{2!}$$
Example 2 (cont.)

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- Because the "first" pair and "second" pair don’t have labels and are indistinguishable from each other.
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\frac{13 \cdot 4 \cdot 12 \cdot 4 \cdot 44 \cdot 1}{2!}
\]

- The other way of getting to the same answer is putting it like this:

\[
\frac{13 \cdot 4 \cdot 4 \cdot 4 \cdot 44 \cdot 1}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 1}
\]

This eliminates the ordering involved in picking the pairs.
**Def:** An *experiment* is something that produces an outcome, which is not predictable with any certainty.

This includes, but is not necessarily restricted to, scientific experiments. Other *experiments* that qualify are:

1. Rolling a die once.
2. Rolling a die $n$ times.
3. Flipping two coins.
4. NL East rankings.
5. Recording the sex of every newborn infant until a male is observed.

One important assumption made is that all possible outcomes are known, which will come up when defining...
**Sample Space**

**Def:** The *sample space* of an experiment is the set of all possible outcomes of an experiment and is denoted by $S$.

Using the previous examples of experiments, the *sample spaces* are:

1. **Rolling a die once** $\iff S = \{1, 2, 3, 4, 5, 6\}$. 
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1. Rolling a die once $\Leftrightarrow S = \{1, 2, 3, 4, 5, 6\}$.
2. Rolling a die $n$ times $\Leftrightarrow S = \{1, 2, 3, 4, 5, 6\}^{100}$. 

(Note: infinite outcomes).
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4. NL East rankings $\iff S = \{\text{all 5! permutations of (ATL, FLA, PHI, NYM, WAS)}\}$. 
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4. NL East rankings ⇔ $S = \{\text{all 5! permutations of (ATL, FLA, PHI, NYM, WAS)}\}$.
5. Recording the sex of every newborn infant until a male is observed ⇔ $S = \{M, FM, FFM, FFFM, \ldots\}$
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4. NL East rankings \( \Leftrightarrow S = \{\text{all 5! permutations of (ATL, FLA, PHI, NYM, WAS)}\} \).
5. Recording the sex of every newborn infant until a male is observed \( \Leftrightarrow S = \{M, FM, FFM, FFFM, \ldots\} \) (Note: infinite outcomes).
6. Lifetime of a transistor \( \Leftrightarrow S = \{x : 0 \leq x < \infty\} \) (Note: continuous outcomes).
Events

- **Def**: An *event* is any subset $E$ of the sample space $E \subset S$.

- Intuitively, an event is a set consisting of possible outcomes of the *experiment*. 
Again, using the previous examples of experiments, possible events are:

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2. Rolling a die $n$ times $\iff$ if $E = \{x \in \{1, 2, 3, 4, 5, 6\}^{100} : \exists$ at least 25 $i$ with $x_i = 3\}$, then $E$ represents at least 25 of the hundred rolls turn up 3.
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3. Flipping two coins ⇔ if $E = \{(HH), (HT)\}$, then $E$ is the event that a head appears on the first coin.
Events (cont.)

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5. Lifetime of a transistor ⇔ if $E = \{x : 0 \leq x \leq 5\}$, then $E$ is the event that the transistor does not last longer than 5 hours.
Example 3

- An executioner offers prisoners on death row a final chance to gain his release.
- The prisoner is to allocate 20 chips, 10 blue and 10 white, into two urns. The division can be made however the prisoner wants, with the only restriction that at least one chip is in a urn.
- The executioner will pick one urn randomly and from that urn picked, the executioner will pick one chip randomly.
- If the chip is white, then the prisoner will be set free. Otherwise, he is *finito*. 
Example 3 (cont.)

- **Question**: What is $S$ in terms of describing the prisoner’s possible allocation options?
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**Solution**: It can be written as

$$S = \{[(1, 0), (9, 10)], [(0, 1), (10, 9)], \ldots, [(10, 9), (0, 1)]\}$$
Example 3 (cont.)

- **Question**: What is $S$ in terms of describing the prisoner’s possible allocation options?

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$$S = \{[(1, 0), (9, 10)], [(0, 1), (10, 9)], \ldots, [(10, 9), (0, 1)]\}$$

- Or more generally as

$$S = \{[(w_1, b_1), (w_2, b_2)] : w_1 + b_1 > 0, w_2 + b_2 > 0, w_1 + w_2 = 10, b_1 + b_2 = 10, w_1, w_2, b_1, b_2 \in (0, 1, 2, \ldots, 10)\}$$
Basic Concepts

- The *union* of two events $A$ and $B$, denoted $A \cup B$, is the event consisting of all outcomes that are either in $A$ or in $B$, or both.

- The *complement* of an event $A$, denoted $A^c$, is the set of all outcomes in $S$ that are not in $A$.

- The *intersection* of two events $A$ and $B$, $A \cap B$, is the event consisting of all outcomes that are in both events.

- When two events $A$ and $B$ have no outcomes in common (i.e. $A \cap B = \emptyset$), $A$ and $B$ are said to be *mutually exclusive*, or *disjoint*, events.
Example 4

- Consider an experiment where we toss a coin 10 times and record the number of heads observed.
- Let $A$ be the event that the number of heads is even, $B$ be the event that the number of heads is odd, and $C$ be $\{6, 7, 8, 9, 10\}$.
- **Question**: What is $A \cup B$?
Example 4

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**Question:** What is $A \cup B$?

**Solution:**

$$A \cup B = \{0, 2, 4, 6, 8, 10\} \cup \{1, 3, 5, 7, 9\} = S$$

**Question:** What is $(A \cap C)^c$?
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- **Solution:**

$$A \cup B = \{0, 2, 4, 6, 8, 10\} \cup \{1, 3, 5, 7, 9\} = S$$

- **Question:** What is $(A \cap C)^c$?
- **Solution:**

$$(A \cap C)^c = (\{0, 2, 4, 6, 8, 10\} \cap \{6, 7, 8, 9, 10\})^c$$
$$= (\{6, 8, 10\})^c = \{0, 1, 2, 3, 4, 5, 7, 9\}$$
These are true for both union and intersection.

**Commutative laws**: \( E \cup F = F \cup E \)

**Associative laws**: \( (E \cup F) \cup G = E \cup (F \cup G) \)

**Distributive laws**: \( (E \cup F) \cap G = E \cap G \cup F \cap G \)

One way of showing this is by *Venn diagrams* (see Figure 2.3, pg 25).
A Venn diagram is a graphical representation for illustrating logical relations among events.
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**Def:** Suppose that there is an experiment with sample space $S$. For each event $E$ in the sample space $S$, define $n(E)$ to be the number of times in the first $n$ repetitions of the experiment that event $E$ occurs. Then $P(E)$, the probability of event $E$, is defined as

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$
A probability is a function $P$: events $\rightarrow \mathbb{R}$ that follows the definition from before and these axioms:

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- **Axiom 2:** $P(S) = 1$. 
A *probability* is a function $P$: events $\rightarrow \mathbb{R}$ that follows the definition from before and these axioms:

- **Axiom 1**: $0 \leq P(E) \leq 1$.
- **Axiom 2**: $P(S) = 1$.
- **Axiom 3**: If $E_i$ are disjoint events (i.e. $E_i \cap E_j = \emptyset$ for any $i \neq j$), then

$$P \left( \bigcup_{1}^{\infty} E_i \right) = \sum_{1}^{\infty} P(E_i)$$
Here are a few of the consequences resulting from the axioms:

1. For any event $E$, we have $E \cap E^c = \emptyset$ and $E \cup E^c = S$. Then

\[ 1 = P(S) = P(E \cup E^c) = P(E) + P(E^c) \]

**Question:** Which axioms?
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**Question:** Which axioms?  
**Solution:** By Axiom 2 and Axiom 3.
Consequences

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**Question:** Which axioms?

**Solution:** By Axiom 2 and Axiom 3.

2. If $E = S$, then $E^c = \emptyset$. More importantly, by the consequence above, we know that

$$P(S^c) = P(\emptyset) = 1 - P(S) = 0$$
4 If $E \subset F$, then $F = E \cup (F \cap E^c)$ and the union is disjoint. Thus, we can use the previous consequence that

$$P(F) = P(E) + P(F \cap E^c) = P(E) + P(F \cap E^c) \geq P(E)$$
Consequences (cont.)

4. If $E \subset F$, then $F = E \cup (F \cap E^c)$ and the union is disjoint. 
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5. The following is also an example of the *inclusion-exclusion principle* we introduce more generally later on. 
Note: $E \setminus F = E \cap F^c$, which is also known as $E$ minus $F$.

$$P(E \cup F) = P(E \setminus F) + P(E \cap F) + P(F \setminus E)$$
Consequences (cont.)

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$$P(E \cup F) = P(E \setminus F) + P(E \cap F) + P(F \setminus E)$$

$$= P(E \setminus F) + P(E \cap F) + P(F \setminus E) + P(E \cap F) - P(E \cap F) \text{ (adding zero)}$$
Consequences (cont.)

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= P(E \setminus F) + P(E \cap F) + P(F \setminus E) \\
+ P(E \cap F) - P(E \cap F) \text{ (adding zero)} \\
= P(E) + P(F) - P(E \cap F)
\]
Example 5

- Some members of a class are members of two groups.
- 50% of the class is a member of group 1, 60% are members of group 2 and 90% are members of at least one group.

**Question**: What fraction of the class is a member of both groups?
Example 5 (cont.)

- Let $S$ be the students in the class, event $E$ be the members of group 1 and event $F$ be the members of group 2.
- Then, from the description, we know $P(E) = .6$, $P(F) = .5$ and $P(E \cup F) = .9$. 
Example 5 (cont.)

Let $S$ be the students in the class, event $E$ be the members of group 1 and event $F$ be the members of group 2.

Then, from the description, we know $P(E) = .6$, $P(F) = .5$ and $P(E \cup F) = .9$.

**Solution:** Then by *inclusion exclusion*, we get that

$$0.9 = P(E \cup F) = P(E) + P(F) - P(E \cap F) = .6 + .5 - P(E \cap F)$$

Thus, $P(E \cap F) = .2$. 
To Do

- Make sure you’ve read 2.4 and begin reading 2.5.
- Check out the homework posted on the course page.
- Make sure June 15th works.