1) (20 points) Suppose you have 4 blue socks and 3 red socks in your drawer (i.e. 7 total socks). In the dark, you draw two socks from your drawer and put them on your feet.

a) What is the chance that you put on matching socks?

\[
P(BB) + P(RR) = \frac{4}{7} \cdot \frac{3}{6} + \frac{3}{7} \cdot \frac{2}{6} = \frac{12 + 6}{42} = \frac{3}{7}
\]

b) What is the chance that you put a red sock on your right foot?

\[\frac{3}{7}\]  (think of Ace on 3rd draw problem)

c) Let \( X \) be the number of blue socks remaining in your drawer. What is the expectation and variance of \( X \)?

\[
X = \begin{cases} 
2 & \text{w.p. } P(BB) = \frac{2}{7} \\
3 & \text{w.p. } P(BR) + P(RB) = \frac{4}{7} \\
4 & \text{w.p. } P(RR) = \frac{1}{7}
\end{cases}
\]

\[
EX = (2) \left( \frac{2}{7} \right) + (3) \left( \frac{4}{7} \right) + (4) \left( \frac{1}{7} \right) = \frac{2.857}{4}
\]

\[
\text{Var} X = \left( 2 - \frac{2.857}{7} \right)^2 \cdot \left( \frac{2}{7} \right) + \left( 3 - \frac{2.857}{7} \right)^2 \cdot \left( \frac{4}{7} \right) + \left( 4 - \frac{2.857}{7} \right)^2 \cdot \left( \frac{1}{7} \right)
\]

\[\approx 0.735 \cdot \left( \frac{2}{7} \right) + (0.020) \cdot \left( \frac{4}{7} \right) + (1.366) \cdot \left( \frac{1}{7} \right) \]

\[= 0.408\]
2) **(10 points)** Consider the following.

a) If one of the terms of the expansion of \((2x - \frac{1}{2}y + 3z - w)^k\) is \((-810) xyz^2w\), then what is \(k\)?

\[
\begin{align*}
x^1 y^2 z^2 w^1 & \implies 1 + 2 + 2 + 1 = \boxed{6 = k} \\
\end{align*}
\]

b) What is the coefficient on the term \(yz^2w^3\), when considering the same expansion from above?

\[
\begin{align*}
\binom{6}{0, 1, 2, 3} \cdot (2)^0 \cdot \left(-\frac{1}{2}\right)^1 \cdot (3)^2 \cdot (-1)^3 \\
= \binom{6}{0} \binom{6}{1} \binom{5}{2} \binom{3}{3} \cdot \left(-\frac{1}{2}\right) (9) \cdot (-1) \\
= \boxed{270}
\end{align*}
\]
3) **(15 points)** The website "BaseballProspectus.com" uses their own model to estimate the probabilities of non-repeatable events involving baseball events. As of Sunday, Baseball Prospectus says that

- The probability that the baseball team the Atlanta Braves, who play in the NL East division, will be the champion of their division is 53%.
- The probability that the Atlanta Braves wins the World Series (i.e. baseball's championship event) is 9%.
- The probability that the NL East champion (i.e. any NL East team that is the champion of their division) will also win the World Series is 17%.

Assume that in order to win the World Series, the Braves or any other NL East team must first become the champions of their division.

a) Given that the Atlanta Braves are the NL East champions, what is the probability that they win the World Series?

\[
P(\text{Atl wins W.S.} \mid \text{Atl wins Div}) = \frac{P(\text{Atl wins Div} \mid \text{Atl wins W.S.}) \cdot P(\text{Atl wins W.S.})}{P(\text{Atl wins Div})} = \frac{(1) \cdot (0.09)}{0.53} = \frac{0.09}{0.53} = 0.17
\]

b) What is the conditional probability that a NL East team wins the World Series, given that the Atlanta Braves are not the NL East champions?

\[
P(\text{Any NL East wins W.S.} \mid \text{Atl loses Div}) = \frac{P(\text{Any NL East wins W.S.} \mid \text{Atl loses Div})}{P(\text{Atl loses Div})} = \frac{(0.17) - (0.09)}{0.47} = \frac{0.08}{0.47} = 0.17
\]
4) **(15 points)** A deck of cards is dealt out.

a) What is the chance that the second and fourth cards are Aces?

\[
\begin{align*}
\frac{50 \cdot 4 \cdot 49 \cdot 3 \cdot 48 \cdots 1}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdots 1} &= \frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{221} \approx 0.0045
\end{align*}
\]

b) What is the expected number of times that an Ace is followed by another Ace?

Let \( A = \# \text{ of times an Ace is followed by an Ace.} \)

\[ A_i = \text{ith card is an Ace and is followed by an Ace.} \]

\[ E(A) = E \left( \sum_{i=1}^{51} A_i \right) = \sum_{i=1}^{51} E(A_i) \]

\[ = (51) \cdot \left( \frac{1}{221} \right) \approx 0.231 \]
5) (10 points) Let $X$ and $Y$ be independent random variables. Suppose that $E(X) = 2$, $E(Y) = -1$, $E(X^2) = 13$ and $\text{Var}(Y) = 4$. Find the $SD(X - 2Y + 8)$.

\[
\text{Var}(X - 2Y + 8) = \text{Var}X + \text{Var}(-2Y)
\]
\[
= \text{Var}X + 4 \text{Var}Y
\]
\[
= 13 - 4
\]
\[
= 9
\]
\[
\Rightarrow 9 + 4(4) = 9 + 16 = 25
\]

Since $\sqrt{\text{Var}} = SD$,

$SD(X - 2Y + 8) = \sqrt{25} = 5$
6) **(15 points)** As a marketing ploy, prizes are put into boxes of cereal. Every cereal box contains a prize. Prize A is in only one fourth of the boxes and prize B is in the remaining boxes. Assume there are an abundance of such boxes.

a) Let $N$ be the number of boxes you need to buy in order to obtain four type A prizes. What are $E(N)$ and $Var(N)$?

\[ N \text{ is negative binomial with } r = 4, \quad p = \frac{1}{4} \]

So

\[ E(N) = \frac{r}{p} = 16 \]

\[ Var(N) = \frac{r(1-p)}{p^2} = (16)(\frac{3}{4}) = 12 \]

b) Suppose you stock the shelves of the local grocery store and you know that the store has exactly 100 boxes of that brand of cereal. You now decide to buy ten boxes of cereal, in hopes of receiving at least nine prizes of type B. Given the marketing ploy is true for this local grocer, what is the probability of that occurring?

Let $X =$ number of type B prizes out of a sample of 10 cereal boxes from a population of 100 boxes.

\[ X \sim \text{hypergeometric with parameters:} \]
\[ N = 100, \quad m = 75, \quad n = 10 \]

So...

\[ P(X \geq 9) = P(X = 9) + P(X = 10) \]

\[ = \binom{75}{9} \binom{25}{1} + \binom{25}{10} \binom{75}{10} = 0.229 \]
7) (15 points) Let \( X \) be a binomial random variable with parameters \( n \) and \( p \). Show that

\[
E \left[ \frac{1}{X+1} \right] = \frac{1 - (1 - p)^{n+1}}{(n+1)p^2}
\]

\[
\frac{1}{X+1} = \begin{cases} 
1 & \text{w.p. } (\frac{8}{6}) p^0 (1-p)^n \\
\frac{1}{2} & \text{w.p. } (\frac{2}{6}) p^1 (1-p)^{n-1} \\
\frac{1}{3} & \text{w.p. } (\frac{1}{6}) p^2 (1-p)^{n-2} \\
\vdots \\
\frac{1}{n+1} & \text{w.p. } (\frac{1}{6}) p^n (1-p)^0 \\
\end{cases}
\]

So...

\[
E \left[ \frac{1}{X+1} \right] = \sum_{i=0}^{n} \frac{1}{i+1} \binom{n}{i} p^i (1-p)^{n-i}
\]

\[
= \sum_{i=0}^{n} \frac{1}{i+1} \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i}
\]

\[
= \sum_{i=0}^{n} \frac{n!}{(n-i)!(i+1)!} p^i (1-p)^{n-i}
\]

Equal to 1: \( n+1 \) comes from

\[
= \sum_{i=0}^{n} \frac{n!}{(n-i)!(i+1)!} \frac{(i+1)!}{p(i+1)!} p^i (1-p)^{n-i}
\]

\[
= \sum_{i=0}^{n} \frac{1}{(n+1)p} \frac{(n+1)!}{(n-i)!(i+1)!} p^i (1-p)^{n-i}
\]

Equal to 1: \( p \) comes from

\[
= \sum_{i=0}^{n} \frac{1}{(n+1)p} \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i}
\]

Knowing \( \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i} = 1 - (1 - p)^{n+1} \) by def of r.v.

Then

\[
E \left[ \frac{1}{X+1} \right] = \frac{1}{(n+1)p} \sum_{i=0}^{n} \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i} = \frac{1}{(n+1)p} \left[ 1 - (1 - p)^{n+1} \right]
\]