Solutions Midterm Exam II - Fall 2004 Statistics 101

1. A. First compute $E(XY) = 1(0.1)+2(0.1)+3(0)+4(0.3)+6(0.3)+9(0.2) = 5.1$. Hence $\text{Cov}(X,Y) = 5.1 - (2.2)(2.2) = 0.26$, and so $\text{Cor}(X,Y) = 0.26/0.46 = 0.565$. Thus ratings tend to go in the same direction; there is a moderate linear relationship between the two ratings.

   B. Plainly the expectation will be the same as the original expectation, viz. 2.2. To compute the standard deviation we use the relationship $\text{Var}(X/2+Y/2) = (1/4)\,\text{Var}(X) + (1/4)\,\text{Var}(Y) + (1/2)\,\text{Cov}(X,Y)$. Thus from the first part we have $\text{Var}(X/2+Y/2) = (1/2).46 + (1/2)0.26 = 0.36$. Thus SD=0.6. Since there is a positive correlation, independence would reduce the overall variance, but have no effect on the mean.

   C. First we compute the conditional probabilities $P(\text{rating} = k|\text{same})$, and these are 0.1/0.6=1/6, 0.3/0.6=1/2, and 0.2/0.6=1/3 for $k = 1, 2, 3$ respectively. Hence $E(\text{rating}|\text{same}) = 1(1/6) + 2(1/2) + 3(1/3) = 2\frac{1}{6}$.

2. A. $P(X > 9) = \int_9^{10} 3x^2/1000dx = 0.271$.

   B. (i) We assume a binomial model. Thus $X \sim B(100, 0.271)$ and $P(20 \le X) = \sum_{k=20}^{100} \binom{100}{k} 0.271^k 0.729^{n-k}$.

   (ii) We use the central limit theorem, $X \approx N(27.1, 27.1(0.729))$, and compute $P(20 \le X) = P((20 - 27.1)/\sqrt(4.4476) \le Z) = P(-1.597 \le Z) = 0.9452$.

   C. (i) OK, integrate x^k to get $x^{k+1}/(k+1)$. Evaluating the definite integral we obtain $10^{k+1}/(k+1)$ and so the normalizing constant is $(k+1)/10^{k+1}$.

   (ii) $E(X) = (k+1)/10^{k+1}) \int_0^{10} x^{k+1}dx = 10(k+1)/(k+2)$. We set this equal to 8 and solve to get $k = 3$.

3. A. Of course it depends on one's stomach for risk. If we consider average risk, then since expected profit is 14(0.2)+(-1)0.8=2>0 we would go for it. However, if we consider downside risk, then since the probability of losing million smackers is 0.8, which is really quite large so we may decide against this venture.

   B. Hopefully we have automatic response to this problem: CLT. Use a normal approximation: the E(profit) on one venture is 2 and the standard deviation is easily worked out to be 6 (lose 1 unit with probability .8 and gain 14 units with probability .2). Thus, since n=25, using the central limit theorem, with $X \sim N(50, 30^2)$ we compute $P(X \le 0) = P(Z \le -50/30) = 0.0475$

   C. (i) Our friend Bayes theorem once again: we want $P(\text{success}|\text{makes 2nd round})$. Displaying the probabilities on a tree diagram or joint table we see that $P(\text{success}|\text{makes 2nd round}) = 1(0.2)/(1(0.2) + 0.5(0.8)) = 1/3$.

   (ii) We want a conditional expectation: $E(\text{profit}|\text{wait}) = 5(1/3)+(-1)2/3 = 1$, less than before so not so smart to wait.

4. A.(i) Putting P into first we have $E(X) = P(8) + (1 - P)(4) \geq 5$ when $P \geq 1/4$.

   (ii) Taking $P = 1/2$, for example, we have Var(CREF/2 + TIAA/2) = Var(CREF)/4 + 0 = $20^2/4 = 100$, and so SD=10.

   B. (i)Of course we need to consider the volatility drag, viz. $\mu - \sigma^2/2$. With the portfolio CREF/2 + TIAA/2, we have $\mu = 0.08/2 + .04/2 = .06$ and $\sigma^2 = .2^2/4 = .01$, and so $\mu - \sigma^2/2 = .06 - .01/2 = .055$.

   (ii) In general we have $\mu - \sigma^2/2 = 0.08P + 0.04(1 - P) - P^2(0.04)/2 = 0.04 + 0.04P - 0.02P^2$. Recalling that a quadratic $ax^2 + bx + c$ has a maximum at $x = -b/(2a)$ we see that $\mu - \sigma^2/2$ has a maximum at -0.04/(-0.04)=1. So put all your dough in the risky investment.

5. A. Here we use our favourite binomial model: We want $P(X = 2) + P(X = 3) = \binom{3}{2}0.3^2(0.7) + 0.3^{3+} = 0.216$.

   B. This one demands a wee bit more concentration, a bit of subtle partitioning. (i) Using the partition rule we have

$$
\begin{aligned}
P(\text{impl}) =& P(\text{impl}|\text{CEO})P(\text{CEO}) + P(\text{impl}|\text{CEO}^c)P(\text{CEO}^c) \\
=& 1(0.3) + 0.5(0.3^2)0.7 \\
=& 0.3315
\end{aligned}
$$

   (This approach has the risk of not counting all the possibilities so here are the possibilities)

CEO	CEO	COO	Probability	Implemented ?	Number
Y	Y	Y	0.027	Y	3
Y	Y	N	0.063	Y	2
Y	N	Y	0.063	Y	2
Y	N	N	0.147	Y	1
N	Y	Y	0.063	HALF	2
N	Y	N	0.147	N	1
N	N	Y	0.147	N	1
N	N	N	0.343	N	0

i) Probability = .027+.063+.063+.147 +.063/2=.3315 as computed above.

ii)

x	p(x)
1	.147/.3315= .4434
2	(.063+.063+.063/2)/.3315= .4751
3	.027/.3315= .0814