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Nonparametric tests for the mean of a non-negative population

Weizhen Wang[a], Linda H. Zhao[b,*]

[a]Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA

[b]Department of Statistics, The Wharton School, 3000 Steinberg Hall-Dietrich Hall, University of Pennsylvania, 3620 Locust Walk, Philadelphia, PA 19104-6320, USA

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Abstract

We construct level-α tests for testing the null hypothesis that the mean of a non-negative population falls below a prespecified nominal value. These tests make no assumption about the distribution function other than that it be supported on $[0,\infty)$. Simple tests are derived based on either the sample mean or the sample product. The nonparametric likelihood ratio test is also discussed in this context. We also derive the uniformly most powerful monotone (UMP) tests for special cases. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

This article concerns the problem of constructing one-sided level-α tests for the population mean of a non-negative random variable. Our discussion thus applies to the common problem of testing the mean survival time based on an uncensored random sample.

Formally, let $X_1,\dots,X_n$ denote the random sample from a population with cumulative distribution F. Assume $P_F(X<0)=0$. In this paper we are primarily interested in constructing nonparametric tests for the one-sided hypothesis

$$\mathrm{H}_0^{(\leqslant)}\colon \mu(F)\leqslant\mu_0,$$

$$\mathrm{H}_a^{(>)}\colon \mu(F)>\mu_0 \text{ or } \mu(F) \text{ does not exist,} \tag{1.1}$$

* Corresponding author.

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where $\mu(F) = E_F(X)$. This is the situation one might encounter for example in trying to establish that the mean survival time with a test treatment exceeds a known baseline value, μ_0.

Let us emphasize that our interest throughout is on examining tests that are valid without any assumptions on F other than the non-negativity assumption $P_F(X < 0) = 0$. Thus we wish to construct level-α tests. These are tests with critical functions ϕ, having power $\pi_\phi(F) = E_F(\phi)$ satisfying

$$\sup\{\pi_\phi(F) \colon F \in \mathrm{H}_0\} \leqslant \alpha. \tag{1.2}$$

We also wish our tests to be informative in the minimal sense that

$$\sup\{\pi_\phi(F) \colon F \in \mathrm{H}_a\} > \alpha. \tag{1.3}$$

Informative level-α tests exist for the problem (1.1).

The simplest informative tests are based on the statistic $\bar{X} = (1/n)\sum_{i=1}^n X_i$ and discussed in the next section. Other simple tests are based on the product statistic $Q = \prod_{i=1}^n X_i$. These tests and some modifications are discussed in Section 3.

An appealing class of tests is those based on the nonparametric likelihood ratio (NPLR). This class of tests is described and commented in Section 4, and we make a conjecture there as to how the NPLR can be used to construct level-α tests. Zhao and Wang (2000) contain further details about NPLR tests and a discussion of a useful family of NPLR tests which are not quite level α.

For the special cases of $n \leqslant 2$ it is possible to derive a UMP monotone test for the problem (1.1). This is done in Section 5. This derivation validates, for $n = 2$, the conjecture made in Section 4 about the NPLR test. It also demonstrates that the level-α tests of Sections 2–4 can all be improved when $n = 2$. We do not believe that a UMP monotone test exists when $n \geqslant 3$, but we feel that the construction in Section 5 nevertheless provides convincing evidence that the tests of Sections 2–4 can be improved for all values of $n \geqslant 2$. Section 6 includes a discussion and comparison of the tests derived in earlier sections. It also contains a plot of the rejection regions of these tests.

There is another important test for this problem that has occasionally been discussed in the literature. Anderson (1967) and Breth, Maritz and Williams (BMW) (1978) describe a test whose foundation is the one-sided Kolmogorov confidence region. A similar construction for a related problem appears in Romano and Wolf (1999). The test proposed by BMW is briefly described in our concluding Section 6.

One qualitative conclusion that can be drawn from the results in our paper is that although tests of (1.1) satisfying (1.2) and (1.3) exist, even the best of them are not very powerful against $F \in \mathrm{H}_a^{(>)}$ unless F is "very far" from $\mathrm{H}_0^{(\leqslant)}$. We comment in more detail about this in Section 6. We note there that all the tests have type I error dramatically less than α over most of $\mathrm{H}_0^{(\leqslant)}$. Overall, BMW's test seems generally preferable to the strictly level-α tests developed in our paper except when n is small.

Apart from their exact level α property none of the tests (including the BMW's test) are very appealing in terms of type I error over $\mathrm{H}_0^{(\leqslant)}$ and power over $\mathrm{H}_0^{(>)}$. This strongly motivates also considering tests that have a nonparametric character but do not strictly satisfy (1.2). Such tests have been considered by various authors. See Owen (1990, 1999), Romano and Wolf (1999), and Zhao and Wang (2000) for such proposals, and other related references.

It is natural to ask why we do not also consider the two-sided problem subject to $P_F(X<0)=0$ of testing

$$\begin{aligned}&\mathrm{H}_0^{(=)}\colon\ \mu(F)=\mu_0,\\&\mathrm{H}_a^{(\neq)}\colon\ \mu(F)\neq\mu_0,\ \text{ or } \mu(F) \text{ does not exist}\end{aligned}\tag{1.4}$$

along with testing of (1.1). Bahadur and Savage (1956) among others have already noted that there is no informative level α-test for this problem when F is not constrained to the non-negative line.

The following proposition directly shows that there is no informative level-α test of

$$\begin{aligned}&\mathrm{H}_0^{(=)}\colon\ \mu(F)=\mu_0,\\&\mathrm{H}_a^{(<)}\colon\ \mu(F)<\mu_0,\ \text{ or } \mu(F) \text{ does not exist,}\end{aligned}$$

where F is supported on $[0,\infty)$. It also can be understood as saying that any test of (1.4) should really be interpreted only as a test of (1.1), since tests of (1.4) can be informative only on $\mathrm{H}_a^{(>)}$. Thus we consider only tests of (1.1).

Proposition 1.1. *For any critical function ϕ, and F^* supported on $[0,\infty)$ and having $\mu(F^*)<\mu_0$*

$$\sup\{\pi_\phi(F)\colon F\in\mathrm{H}_0^{(=)}, P_F(X<0)=0\}\geqslant\pi_\phi(F^*).\tag{1.5}$$

Consequently, if ϕ defines a level-α test of $\mathrm{H}_0^{(=)}$ then

$$\pi_\phi(F)\leqslant\alpha\quad\textit{whenever } \mu(F)<\mu_0.\tag{1.6}$$

Proof. Define $F_\gamma\in\mathrm{H}_0^{(=)}$ by $F_\gamma=(1-\gamma)F^*+\gamma I_{[\mu(F^*)+\frac{\mu_0-\mu(F^*)}{\gamma},\infty)}$. Then $\mu(F_\gamma)=(1-\gamma)\mu+\gamma(\mu(F^*)+\frac{\mu_0-\mu(F^*)}{\gamma})=\mu_0$, as desired. Also $\pi_\phi(F_\gamma)\to\pi_\phi(F^*)$ as $\gamma\to 0$ for any critical function ϕ. This yields (1.5). Assertion (1.6) follows logically from (1.5). □

2. Tests based on the sample mean

With no loss of generality we assume throughout the remainder of the paper that $\mu_0=1$. The simplest informative level-α test for the one-sided problem (1.1) is given by

$$\phi_{1/\alpha}(X_1,\ldots,X_n)=\begin{cases}1 & \text{if } \bar{X}\geqslant 1/\alpha,\\ 0 & \text{if } \bar{X}<1/\alpha.\end{cases}\tag{2.1}$$

This test has level α as a consequence of the elementary Markov inequality. That is, for $F \in \mathrm{H}_0^{(\leqslant)}$

$$\begin{aligned}\pi_{\phi_{1/\alpha}} &= P_F(\bar{X} \geqslant 1/\alpha)\\ &\leqslant \frac{1}{1/\alpha} E_F(\bar{X}) \leqslant \alpha. \end{aligned} \tag{2.2}$$

This test is also informative in our sense in that

$$\sup\{\pi_{\phi_{1/\alpha}}(F)\colon F \in \mathrm{H}_a^{(>)}\} = 1 > \alpha. \tag{2.3}$$

(Of course, $\inf\{\pi_{\phi_{1/\alpha}}(F\colon F \in \mathrm{H}_a^{(>)}\} = 0 \leqslant \alpha$. This is an inevitable nondesirable property of any reasonable level-α test of (1.1).)

For $n = 1$ the Markov inequality is sharp in the sense that there is a distribution F having $\mu(F) = 1$ for which equality holds in (2.2). For $n \geqslant 2$ the inequality is not sharp. Hoeffding and Shrikhande (1955), building on Birnbaum et al. (1947) establish that for $c \geqslant 2$, $n \geqslant 2$ and $\mu(F) = 1$

$$P_F(\bar{X} \geqslant c) \leqslant \begin{cases} \frac{1}{c} - \frac{1}{4c^2} & \text{if } n \text{ is even,} \\ \frac{1}{c} - \frac{n^2-1}{n^2}\frac{1}{4c^2} & \text{if } n \text{ is odd.} \end{cases} \tag{2.4}$$

They also point out the lower bound

$$\sup\{P_F(\bar{X} \geqslant c)\colon \mu(F) \leqslant 1\} \geqslant 1 - \left(1 - \frac{1}{nc}\right)^n. \tag{2.5}$$

Samuels (1969) proves the lower bound above is sharp when $c \geqslant \max(4, n-1)$. Beginning from results in Samuels (1969) we give in the following theorem an upper bound of $\sup\{P_F(\bar{X} \geqslant c)\colon \mu(F) = 1\}$ which is very close to that in (2.5).

Theorem 2.1. *Let* $[c]$ *denote the largest integer less than or equal to c. Let* $c' = \min\{[c]+1, n\}$ *and* $\Lambda(n,c) = n \bmod(c')$, *where* $0 \leqslant \Lambda(n,c) < c'$. *Then when* $c \geqslant 4$ *and* $n \geqslant 5$

$$\begin{aligned}\sup\{P_F(\bar{X} \geqslant c)\colon \mu(F) = 1\} &\leqslant U(n,c)\\ &\stackrel{\text{def}}{=} 1 - \left(1 - \frac{[n/c']}{nc}\right)^{c'-\Lambda}\left(1 - \frac{[n/c']+1}{nc}\right)^{\Lambda}\\ &\leqslant \frac{1}{c}. \end{aligned} \tag{2.6}$$

See the appendix for a proof.

Remark 1. When $c \geqslant n-1$, $U(n,c) = 1 - (1 - 1/nc)^n$ which is the lower bound given in (2.5). The result is obtained in Samuels (1969).

Remark 2. When $c < n - 1$, $U(n,c)$ is most easily interpreted when both c and n/c are integers. In this case

$$U(n,c) = 1 - \left(1 - \frac{1}{c(c+1)}\right)^{c+1} = \frac{1}{c} - \frac{1}{2c^2} + O\left(\frac{1}{c^3}\right). \tag{2.7}$$

This can be compared to the right-hand side of (2.4). It is evident that when c is a moderate to large number then (2.4) improves on the bound (2.4), and is close to the best conceivable inequality since also $(1 - 1/nc)^n = 1/c - 1/2c^2 + O(1/c^3)$.

Bounds (2.6) and (2.4) can be used to define a level-α test which is better than $\phi_{1/\alpha}$. Let c_6 satisfy

$$U(n, c_6) = \alpha. \tag{2.8}$$

Then ϕ_{c_6} has level α. For an algebraically simpler but slightly inferior test, one can instead use (2.4) to get that ϕ_{c_4} has level α where c_4 solves

$$\alpha = \begin{cases} \frac{1}{c} - \frac{1}{4c^2} & \text{if } n \text{ is even,} \\ \frac{1}{c} - \frac{n^2-1}{n^2}\frac{1}{4c^2} & \text{if } n \text{ is odd.} \end{cases} \tag{2.9}$$

In particular, if n is even then $c_4 = [1 + \sqrt{(1-\alpha)}/2\alpha]$.

It is of interest to compare c_6, c_4 to c_5 where

$$1 - \left(1 - \frac{1}{nc_5}\right)^n = \alpha. \tag{2.10}$$

(According to (2.5) c_5 provides the lower bound for all the tests of the form ϕ_c which are α level.)

Table 1 gives critical values of $c_1 = 1/\alpha$, c_4, c_6 and c_5 for selected choices of α and n. Note that c_5 is of course smaller than all others, but the differences are not large. Also c_6 is very close to c_5 and is less than c_4 throughout the table.

As an alternative approach, the one-sided t-statistic is often used to test (1.1). However, the resulting test is not level α. In fact it is of size one, as formally shown by the following proposition. For this let $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

Table 1
The critical values of c_6, c_4 and the lower bound c_5

α	$c_1 = 1/\alpha$	$n = 10$		$n = 50$		$n = \infty$		n even
		c_6	c_5	c_6	c_5	c_6	c_5	c_4
0.2	5	4.58	4.53	4.58	4.49	4.48	4.48	4.74
0.1	10	9.54	9.54	9.54	9.50	9.49	9.49	9.74
0.05	20	19.55	19.55	19.52	19.51	19.50	19.50	19.75
0.01	100	99.55	99.55	99.51	99.51	99.5	99.5	99.75

Proposition 2.1. *For problem* (1.1), *the traditional t-test*, *which rejects the null hypothesis if*

$$\frac{\bar{X}-1}{S/\sqrt{n}} > t_{\alpha,n-1}, \tag{2.11}$$

where $t_{\alpha,n-1}$ *is the upper* α *quantile of Student-t distribution with* $n-1$ *degrees of freedom*, *has size one.*

Proof. Let P_k be the probability measure on two points 0 and $b_k = 1 + 1/k$ with the probabilities, $1 - b_k^{-1}$ and b_k^{-1}, respectively. So the mean of P_k is one and b_k^{-n} goes to one as k goes to infinity. It is clear that the sample point $\tilde{X}_k = (b_k, \ldots, b_k)_{1\times n}$ belongs to the rejection region (2.11) for any k. Therefore, the size of the test is at least $\lim_{k\to\infty} P(\{\tilde{X}_k\}) = \lim_{k\to\infty} b_k^{-n} = 1$. □

3. Tests based on the sample product

Another type of level-α tests of (1.1) is based on the sample product. Consider the critical function

$$\xi_c(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \prod_{i=1}^n X_i \geqslant c, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Theorem 3.1. *When* $c = 1/\alpha$, ξ_c *is a size* α *test. That is*, $\xi_{1/\alpha}$ *satisfies*

$$\sup\{\pi_{\xi_{1/\alpha}}(F)\colon \mu(F) \leqslant 1\} = \alpha. \tag{3.2}$$

Proof. Notice that for a distribution F with $\mu(F) \leqslant 1$,

$$\begin{aligned} \pi_{\xi_{1/\alpha}}(F) &= P\left(\prod_{i=1}^n X_i \geqslant 1/\alpha\right) \\ &\leqslant \alpha E\left(\prod_{i=1}^n X_i\right) = \alpha \prod_{i=1}^n EX_i \leqslant \alpha. \end{aligned} \tag{3.3}$$

Also, if we choose

$$F_0 = (1 - \alpha^{1/n}) I_{[0,\infty)} + \alpha^{1/n} I_{[1/\alpha^{1/n},\infty)} \tag{3.4}$$

then $\mu(F_0) = 1$ and

$$\pi_{\xi_{1/\alpha}}(F_0) = P\left(\prod X_i \geqslant 1/\alpha\right) = P(X_i = 1/\alpha^{1/n},\ i = 1, \ldots, n) = \alpha. \tag{3.5}$$

This proves (3.2). □

Because of (3.5) no test of the form ξ_c with $c < 1/\alpha$ can have level α. So ξ_c cannot be improved as a level-α test by reducing its critical value, as was the case with ϕ_c in the previous section. However, it is possible to describe uniformly more powerful tests than $\xi_{1/\alpha}$ which have larger rejection regions but still have level α. Here is one such test.

Theorem 3.2. *Let* $d^* = [1-(1-\alpha)^{1/n}]^{-1}$ *and*

$$\xi^*(X_1,\dots,X_n) = \begin{cases} 1 & \text{if } \max(X_i) \geqslant d^* \text{ or if } \xi_{1/\alpha}(X_1,\dots,X_n)=1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Then

$$\pi_{\xi^*}(F) \geqslant \pi_{\xi_{1/\alpha}}(F) \tag{3.7}$$

with strict inequality for some distributions, F. Furthermore, $\xi^*(x)$ *is a level-*α *test.*

The proof is given in the appendix.

4. The NPLR test

The nonparametric likelihood ratio (NPLR), Λ, is defined as follows. Let $\mathscr{D}$ denote the collection of discrete distributions on $[0,\infty)$ and let $\mathscr{D}^{(\leqslant)} = \{F\colon F \in \mathscr{D} \cap \mathrm{H}_0^{(\leqslant)}\}$. For any $(X_1,\dots,X_n) \in [0,\infty)^n$ the nonparametric likelihood at $F \in \mathscr{D}$ is

$$L(F;X_1,\dots,X_n) = \prod_{i=1}^{n} F(\{X_i\}). \tag{4.1}$$

Then

$$\Lambda(X_1,\dots,X_n) = \frac{\sup\{L(F;X_1,\dots,X_n)\colon F \in \mathscr{D}\}}{\sup\{L(F;X_1,\dots,X_n)\colon F \in \mathscr{D}^{(\leqslant)}\}}. \tag{4.2}$$

With this definition of Λ large values of the NPLR lead to rejection of $\mathrm{H}_0^{(\leqslant)}$.

Motivation for this test can be found in Dvoretzky et al. (1956, Sections 5–7) and Kiefer and Wolfowitz (1956). For more recent discussions and several additional references consult Owen (1990, 1999).

Theorem 4.1. *Assume* $X_i \neq X_j$ *for* $1 \leqslant i < j \leqslant n$. *Then*

$$\Lambda(X_1,\dots,X_n) = \sup_{0\leqslant\rho\leqslant 1} \prod_{i=1}^{n} (1+\rho(X_i-1)). \tag{4.3}$$

Note that when $\rho = 1$

$$\prod_{i=1}^{n} (1+\rho(X_i-1)) = \prod_{i=1}^{n} X_i.$$

Hence,

$$\Lambda(X_1,\dots,X_n) \geqslant \prod_{i=1}^{n} X_i. \tag{4.4}$$

It can also be easily checked that equality holds in (4.4) *if and only if*

$$\sum_{i=1}^{n} X_i^{-1} \leqslant n. \tag{4.5}$$

The proof is deferred to the appendix.

Consider the test with critical function

$$\eta_{1/\alpha}(X_1,\ldots,X_n) = I_{\{\Lambda(X_1,\ldots,X_n)\geqslant 1/\alpha\}}. \tag{4.6}$$

Note that $\eta_{1/\alpha} \geqslant \xi_{1/\alpha}$ because of (4.4), with strict inequality for some $(X_1,\ldots,X_n)$. Hence, $\eta_{1/\alpha}$ is uniformly more powerful than $\xi_{1/\alpha}$. We conjecture that $\eta_{1/\alpha}$ is level α nevertheless — i.e., that

$$\sup\{\pi_{\eta_{1/\alpha}}(F)\colon \mu(F) \leqslant 1\} = \alpha. \tag{4.7}$$

When $n = 2$ one can check that

$$\Lambda = \begin{cases} 1 & \text{if } \bar{X} \leqslant 1, \\ X_1X_2 & \text{if } \bar{X} > 1,\ X_1^{-1} + X_2^{-1} \leqslant 2, \\ -\frac{(X_2-X_1)^2}{4(X_2-1)(X_1-1)} & \text{if } \bar{X} > 1,\ X_1^{-1} + X_2^{-1} > 2. \end{cases} \tag{4.8}$$

The results of Section 5 then shows that this conjecture is true (i.e. $\eta_{1/\alpha}$ is level α) when $n = 2$.

For $n \geqslant 3$ we have not been able to prove or disprove this conjecture. (We have checked that (4.7) holds for specific choices of $n \geqslant 3$ and for a variety of simple discrete distributions for F.) Zhao and Wang (2000) investigate a class of tests based on Λ which, however, are not level α over all of $\mathrm{H}_0^{(\leqslant)}$.

5. The UMP test for $n \leqslant 2$

In this section, we provide the uniformly most powerful (UMP) tests in certain test classes when the sample size is less than or equal to two.

5.1. $n = 1$

When $n = 1$, the uniformly most powerful nonrandomized test exists. Note that the tests $\phi_{1/\alpha}$ in (2.1), $\xi_{1/\alpha}$ in (3.1), and $\eta_{1/\alpha}$ in (4.6) coincide. We have the following result.

Theorem 5.1. *When $n=1$, $\phi_{1/\alpha}(x)=I_{\{x\geqslant 1/\alpha\}}$ is the UMP level-α test among all level-α nonrandomized tests.*

Proof. Eq. (2.2) shows that $\phi_{1/\alpha}$ is a level α test. For any nonrandomized level-α test ϕ, it suffices to show

$$\phi(x) \leqslant \phi_{1/\alpha}(x). \tag{5.1}$$

Suppose the above is not true. Then there is a point $x_0 < 1/\alpha$ so that $\phi(x_0) = 1 > 0 = \phi_{1/\alpha}(x_0)$. If $x_0 < 1$, let P_0 be a point probability measure on x_0. Then $P_0 \in \mathrm{H}_0^{(\leqslant)}$ and ϕ is not a level-α test because of $E_{P_0}\phi(x) = 1 > \alpha$, a contradiction; If $x_0 \geqslant 1$, let P_0 be a measure having masses $1 - 1/x_0$ and $1/x_0$ at zero and x_0, respectively. Again, $P_0 \in \mathrm{H}_0^{(\leqslant)}$ and $E_{P_0}\phi(x) = 1/x_0 > \alpha$, a contradiction. □

There exists a uniformly more powerful test than $\phi_{1/\alpha}$. Let

$$k_1(X) = \min(\alpha X,\ 1).$$

This function takes all values between 0 and 1, and then defines a randomized test. It is obvious that $k_1(x) \geqslant \phi_{1/\alpha}(x)$ and the strict inequality holds when $x \in (0, 1/\alpha)$. Also

$$E_P(k_1(X)) \leqslant \alpha E_P(X) \leqslant \alpha$$

for any probability $P \in \mathrm{H}_0^{(\leqslant)}$. Thus, k_1 is level-α and strictly more powerful than $\phi_{1/\alpha}$ whenever $P(0 < X < 1/\alpha) > 0$. A natural question is then raised: Does a UMP test among all tests exist? Here is a negative answer.

Proposition 5.1. *When $n = 1$ and $0 < \alpha < 1$ there is no UMP level-α test among all level-α tests.*

Proof. Suppose a UMP level-α test $k_2(X)$ exists.

First, we show that

$$k_2(x) = k_1(x)$$

when $x > 1$. If this is not true, then there exists a point x_0 so that (i) $k_2(x_0) > k_1(x_0)$ or (ii) $k_2(x_0) < k_1(x_0)$. For case (i), since $k_1(x) = 1$ when $x \geqslant 1/\alpha$, $x_0 \in (1, 1/\alpha)$. Let P_2 be a measure having two masses $1 - 1/x_0$ and $1/x_0$ at zero and x_0, respectively. Then $P_2 \in \mathrm{H}_0^{(\leqslant)}$ and

$$E_{P_2}k_2(X) \geqslant k_2(x_0)\frac{1}{x_0} > k_1(x_0)\frac{1}{x_0} = \alpha,$$

which implies that $k_2(x)$ is not level α, a contradiction. For case (ii), let P_3 be a point probability at x_0. Then $P_3 \in \mathrm{H}_a^{(>)}$ and

$$E_{P_3}k_2(X) = k_2(x_0) < k_1(x_0) = E_{P_3}k_1(X).$$

This contradicts the fact that $k_2(X)$ is a UMP test.

Secondly, we show that $k_2(x) = k_1(x)$ on [0,1] as well. For any $x_0 \in [0, 1]$, consider a probability P_4 having two masses $(1 - \alpha)/(1 - x_0\alpha)$ and $(\alpha - \alpha x_0)/(1 - x_0\alpha)$ at x_0 and $1/\alpha$, respectively. It is easy to check that $E_{P_4}X = 1$. Since a UMP test must be a similar test in this problem, $E_{P_4}k_2(X) = \alpha$, which implies

$$k_2(x_0) = \alpha x_0 = k_1(x_0).$$

So far we have proved that the UMP test is equal to $k_1(x)$, provided it exists. Consider a probability P_5 having two masses $1 - \alpha/2$ and $\alpha/2$ at zero and $3/\alpha$, respectively.

$P_5 \in \mathrm{H}_a^{(>)}$. However,

$$E_{P_5} k_1(X) = \alpha/2 < \alpha.$$

This contradicts with the fact that a UMP test always has a power at least α. Therefore, no UMP test exists. □

5.2. $n = 2$

When $n = 2$, for a sample X_1, X_2 from distribution P on $[0, \infty)$, let $X_{(1)}, X_{(2)}$ denote the ordered values of X_1, X_2. Define $\mathscr{P}_0 = \{P \text{ on } [0, \infty)\colon E_P(X) \leqslant 1\}$. Test H_0: $P \in \mathscr{P}_0$ versus H_a: $P \in \{P \text{ on } [0, \infty), P \notin \mathscr{P}_0\}$. Let ϕ be a test function. We say ϕ is strongly monotone if ϕ is a symmetric function such that $x < x', y < y'$ and $\phi(x, y) > 0 \Rightarrow \phi(x', y') = 1$; $x < x', y < y'$ and $\phi(x', y') < 1 \Rightarrow \phi(x, y) = 0$. We will give the UMP strongly monotone test for $n = 2$.

Fix α. Let $\hat{T} = \frac{1+\sqrt{1-\alpha}}{\alpha} = (1 - \sqrt{1-\alpha})^{-1}$. Define $s(t)$ by

$$s(t) = \begin{cases} t - \sqrt{t^2 - 1/\alpha} & \text{if } \frac{1}{\sqrt{\alpha}} \leqslant t \leqslant \frac{1}{\alpha}, \\ t - \frac{t-1}{\sqrt{1-\alpha}} & \text{if } \frac{1}{\alpha} < t \leqslant \hat{T}. \end{cases} \tag{5.2}$$

Then for $s \leqslant t$ define

$$\phi^*(s,t) = \phi^*(t,s) = \begin{cases} 1 \text{ if } t \geqslant \hat{T}, \\ 1 \text{ if } s \geqslant s(t), \frac{1}{\sqrt{\alpha}} \leqslant t < \hat{T}, \\ 0 \text{ otherwise.} \end{cases} \tag{5.3}$$

Theorem 5.2. *When $n = 2$, the test $\phi^*(X_1, X_2)$ defined by (5.3) is the UMP strongly monotone level-α test. In fact, if ϕ is any other strongly monotone level-α test then*

$$\phi \leqslant \phi^*.$$

The proof is deferred to the appendix.

6. Discussion

In Sections 2 and 3, we constructed level-α tests of $\mathrm{H}_0^{(\leqslant)}$ versus $\mathrm{H}_a^{(>)}$. For $n = 2$ these tests are strongly monotone and none of these tests is ϕ^* defined by (5.3) which is the same to that in (7.24). Hence for $n = 2$ all of the tests defined in Sections 2 and 3 can be improved by the level-α test ϕ^*.

It can also be checked that the NPLR test, $\eta_{1/\alpha}$, satisfies

$$\xi_{1/\alpha}(x_1, x_2) \leqslant \eta_{1/\alpha}(x_1, x_2) \leqslant \xi^*(x_1, x_2) \leqslant \phi^*(x_1, x_2) \tag{6.4}$$

with each inequality being strict for some values of x_1, x_2. As a consequence of (6.4) it follows for $n = 2$ that $\eta_{1/\alpha}$ is level α for testing $\mathrm{H}_0^{(\leqslant)}$, but that it is less powerful than ξ^* and ϕ^*.

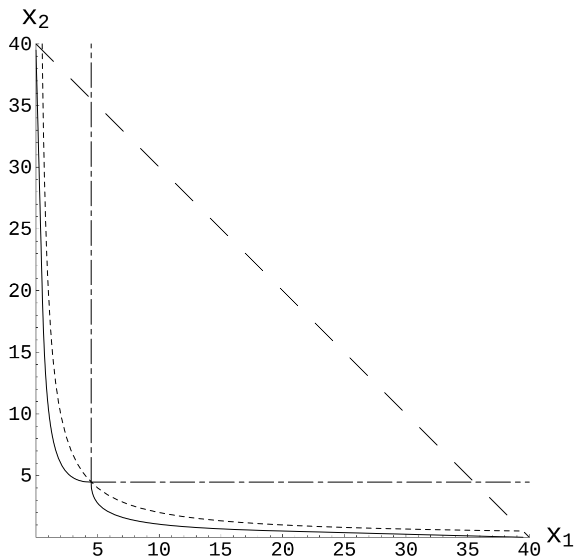

Fig. 1. Boundaries of rejection regions for $\alpha = 0.05$. Solid line: the UMP test ϕ^*; dot line: ξ^*; dashed line: ϕ_{c_6}; and dashed-dot line: BMW test.

Fig. 1 shows the boundaries of the critical regions for $\alpha = 0.05$ of the level-α tests of the form ϕ_{c_6} (in this case, $c_6 = c_5$), the best test based on a sum; ξ^*, the improved product-based test; and ϕ^*, the UMP strongly monotone test. One can see, as already noted that ϕ^* is more powerful than each of the other tests but between ξ^* and ϕ_{c_6} neither one dominates the other. Nevertheless, visually the test ξ^* appears to have a much more desirable rejection region of the two, and to be not much worse than ϕ^*.

The boundary of $\eta_{1/\alpha}$, the NPLR test, is not shown in Fig. 1. It coincides with that of ξ^* along that boundary over the range $\max(x_1, x_2) < \hat{T} = 39.49$. Then it continues as a curve sandwiched strictly between $x_1x_2 = 20$ (the boundary of $\xi_{1/\alpha}$) and the coordinate axes.

Breth et al. (1978) proposed a different test, as follows. Let $k(\varepsilon) = [n - n\varepsilon]$ and let $\varepsilon = \varepsilon_\alpha$ solve

$$\alpha = \varepsilon \sum_{j=0}^{k(\varepsilon)} \binom{n}{j} \left(\varepsilon + \frac{j}{n}\right)^{j-1} \left(1 - \varepsilon - \frac{j}{n}\right)^{n-j}. \tag{6.5}$$

Then let $X_{(1)} \leqslant X_{(2)}, \ldots, \leqslant X_{(n)}$ denote the ordered $X_1, X_2, \ldots, X_n$ and let $k_\alpha = k(\varepsilon_\alpha)$ and

$$H = \frac{1}{n} \left\{ \sum_{i=1}^{k_\alpha} X_{(i)} + (n - k_\alpha - n\varepsilon_\alpha) X_{(k_\alpha+1)} \right\}. \tag{6.6}$$

Reject $\mathrm{H}_0^{(\leqslant)}$ when $H > 1$. H can be most easily described when $n - n\varepsilon$ is an integer. In this case, take the average of the k_α smallest order statistics and $n - k_\alpha$ zeros; reject if this average exceeds 1. If $[n-n\varepsilon]<k_\alpha<[n-n\varepsilon]+1$ then a portion of $X_{(k_\alpha+1)}$ must be included in this average.

As BMW note it is easy to see that this test has level α. Let F_n denote the sample CDF from $X_1,\ldots,X_n$ and let $F^*_{n,\varepsilon}(t)=\min\{F_n(t)+\varepsilon,1\}$. Then Wilks (1962, p. 337) yields

$$1-\alpha\leqslant P_F\{F(t)\leqslant F^*_{n,\varepsilon}(t) \text{ for all } t\}.$$

Hence for $F\in \mathrm{H}_0^{(\leqslant)}$

$$1-\alpha\leqslant P_F\left\{1\geqslant\int_0^\infty t\,\mathrm{d}F(t)=\int_0^\infty(1-F(t))\,\mathrm{d}t\geqslant\int_0^\infty(1-F^*_{n,\varepsilon}(t))\,\mathrm{d}t\right.$$
$$\left.=\int_0^\infty t\,\mathrm{d}F^*_{n,\varepsilon}(t)=H\right\}.$$

Now we provide further discussions on tests BMW, ξ^* and ϕ_{c_6}. When $n=2$ then $\varepsilon_\alpha=1-\sqrt{\alpha}$. So for $\alpha=0.05$, $\varepsilon_{0.05}=0.776$. Consequently for $n=2$ the BMW's test rejects whenever

$$\min(X_1,X_2)\geqslant\frac{1}{\sqrt{\alpha}}=4.47. \tag{6.7}$$

As shown in Fig. 1, when $n=2$ this test is dominated by ξ^*. Neither of ϕ_{c_6} nor BMW's test dominates the other, although BMW's test might appear more generally desirable.

As n increases BMW's test quickly appears to be generally the most favorable relative to ξ^* and ϕ_{c_6} though none of the three tests dominates any other. A brief example can make this clear. More importantly it also will show that none of the tests has appealing behavior overall, apart from being level α.

BMW consider the case of $F_{\exp}(t)=1-\mathrm{e}^{-t}\in\mathrm{H}_0^{(\leqslant)}$ and $n=5$ and $\alpha=0.1$. They calculate that $\varepsilon=0.447$ and the type I error of the test at $F_{\exp}$ is

$$P_{F_{\exp}}\{H>1\}=0.001<0.01.$$

Both ξ^* and ϕ_{c_6} have even less type I error at $F_{\exp}$ by several orders of magnitude. On the other hand, consider $n=5$, $\alpha=0.1$ and $F=G_{50}=\frac{49}{50}I_{[0,\infty)}+\frac{1}{50}I_{[50,\infty)}$. Then $F\in\mathrm{H}_0^{(\leqslant)}$ and $0.1=P_{G_{50}}\{\phi_{c_6}\ \textit{rejects}\}$, but the other two tests have smaller type I error. Finally, if $n=5$, $\alpha=0.1$ and $F=G_{10^{1/5}}=(1-10^{-1/5})I_{[0,\infty)}+10^{-1/5}I_{[10^{1/5},\infty)}\in\mathrm{H}_0^{(\leqslant)}$ then $0.1=P_{G_{10^{1/5}}}\{\xi^*\ \textit{rejects}\}$ but $P_{G_{10^{1/5}}}\{\phi_{c_6}\ \textit{rejects}\}=0=P\{\textit{BMW rejects}\}$.

The above pattern is typical for all but very small n. It can for such n be summarized that for unimodal and similarly well-behaved $F\in\mathrm{H}_0^{(\leqslant)}$ BMW's test has type I error much less than α but much larger than ξ^* or ϕ_{c_6}. On the other hand, no test dominates the others, and there are some very particular distributions in $\mathrm{H}_0^{(\leqslant)}$ under which either ξ^* or ϕ_{c_6} has type I error $=\alpha$, and is very much better than the other two tests. It should be clear from the fact that the type I error of these tests is so small on $\mathrm{H}_0^{(\leqslant)}$ that they have useful power only when $F\in\mathrm{H}_a^{(>)}$ is rather far from $\mathrm{H}_0^{(\leqslant)}$.

Appendix A

Proof of Theorem 2.1. Samuels (1969) proves that if $Y_1,\ldots,Y_m$ are independent non-negative random variables with mean v_i and $S_m=\sum_{i=1}^m Y_i$ then

$$P(S_m \geqslant mt) \leqslant 1-\prod_{i=1}^m\left(1-\frac{v_i}{mt}\right), \tag{A.1}$$

so long as $mt > \max[4,(m-1)]\sum_{i=1}^m v_i$. The same result for $mt = \max[4,(m-1)]\sum_{i=1}^m v_i$ follows by continuity in $\{v_i\}$.

For $c > n-1 \geqslant 4$, $U(n,c)=1-(1-1/nc)^n$. In this situation, Samuels (1969) proves the validity of (2.6).

For $4 \leqslant c \leqslant n-1$, define

$$k_j=\begin{cases} j\left[\frac{n}{c'}\right] & \text{if } j=0,1,\ldots,c'-\Lambda,\\ j\left[\frac{n}{c'}\right]+j-(c'-\Lambda) & \text{if } j=c'-\Lambda+1,\ldots,c'. \end{cases}$$

If $\Lambda=0$ then $k_j=j[n/c']$ for $j=0,\ldots,c'$. In particular, $k_0=0$, $k_1=[n/c']$, and $k_{c'}=n$. Also note that

$$n=c'[n/c']+\Lambda.$$

Define

$$Y_j=\sum_{i=k_{j-1}+1}^{k_j} X_i, \quad j=1,\ldots,c'.$$

Then

$$E(Y_j)=\begin{cases} [n/c'] & \text{if } j \leqslant c'-\Lambda,\\ [n/c']+1 & \text{if } c'-\Lambda+1 \leqslant j \leqslant c'. \end{cases}$$

Then from (A.1) we have

$$\begin{aligned} P(\bar{X} \geqslant c) &= P\left(\sum_{j=1}^{c'} Y_j \geqslant nc\right)\\ &\leqslant 1-\left(1-\frac{[n/c']}{nc}\right)^{c'-\Lambda}\left(1-\frac{[n/c']+1}{nc}\right)^{\Lambda}\\ &= U(n,c). \end{aligned}$$

Finally, note that $U(n,c) < 1/c$ since $(1-a)^b > 1-ab$ for $0<a<1$, $b \geqslant 2$. $\square$

Proof of Theorem 3.2. Let $\mu(F) \leqslant 1$ and let $\rho_1 = P_F(X_1 \geqslant d^*)$. Note that $0 \leqslant \rho_1 \leqslant 1/d^* = 1-(1-\alpha)^{1/n}$. Then

$$\pi_{\xi^*}(F) = P_F(\max X_i \geqslant d^*) + P(\max X_i < d^*)P\left(\prod X_i \geqslant 1/\alpha \,|\, \max X_i < d^*\right). \tag{A.2}$$

Now note that

$$P_F(\max X_i \geqslant d^*) = 1 - (1-\rho_1)^n. \tag{A.3}$$

Also

$$E(X_i|X_i < d^*) = \frac{\int_0^\infty x\,\mathrm{d}F(x) - \int_{d^*}^\infty x\,\mathrm{d}F(x)}{1-\rho_1} \leqslant \frac{1-\rho_1 d^*}{1-\rho_1}, \tag{A.4}$$

then

$$P\left(\prod X_i \geqslant 1/\alpha | \max X_i < d^*\right) \leqslant \left(\frac{1-\rho_1 d^*}{1-\rho_1}\right)^n \alpha. \tag{A.5}$$

Combining (A.2)–(A.5) yields

$$\begin{aligned}\pi_{\xi^*}(F) &\leqslant 1-(1-\rho_1)^n + (1-\rho_1)^n\left(\frac{1-\rho_1 d^*}{1-\rho_1}\right)^n \alpha \\ &= 1-(1-\rho_1)^n + (1-\rho_1 d^*)^n\alpha \stackrel{\mathrm{def}}{=} g(\rho_1). \end{aligned} \tag{A.6}$$

Note that $\rho_1 \leqslant 1-(1-\alpha)^{1/n} = 1/d^*$,

$$g(0) = \alpha = g(1-(1-\alpha)^{1/n}), \tag{A.7}$$

$$g'(\rho_1) = n(1-\rho_1)^{n-1} - nd^*\alpha(1-\rho_1 d^*)^{n-1}. \tag{A.8}$$

Hence,

$$g'(0) = n - nd^*\alpha < 0 \tag{A.9}$$

and

$$g'(1/d^*) > 0. \tag{A.10}$$

Finally, $g'(\rho)$ is continuous and has a unique root in $(0, 1/d^*)$. (This root is $\frac{(d^*\alpha)^{1/(n-1)}-1}{(d^*\alpha)^{1/(n-1)}d^*-1}$.) It follows from (A.7)–(A.10) that

$$g(\rho_1) \leqslant \alpha \tag{A.11}$$

(with equality only for $\rho_1 = 0$ and $\rho_1 = 1/d^*$). Refer this fact to (A.6) to see that $\pi_{\xi^*}(F) \leqslant \alpha$ whenever $\mu(F) \leqslant 1$. □

Proof of Theorem 4.1. It is trivial that the supremum in the numerator of (4.2) occurs when $F=F_1$, where $F_1(\{X_i\})=1/n$, $i=1,\ldots,n$. It is also easy to see that the supremum in the denominator of (4.2) must occur at some F for which $F(\{X_i\}) > 0$, $i=1,\ldots,n$, and $F(\{t\})=0$ unless $t=0$ or X_i for some $i=1,\ldots,n$. Hence, let $a_i=F(\{X_i\})$, $i=1,\ldots,n$. If $\min(X_i) > 0$ then let $a_0 = F(\{0\})$; otherwise, set $a_0 = 0$. Note that

$$a_i \geqslant 0, \ \sum_{i=0}^n a_i = 1, \tag{A.12}$$

$$\sum a_i X_i \leqslant 1. \tag{A.13}$$

The supremum in the denominator of (4.2) can then be rewritten

$$\sup\{L(F;\ X_1,\ldots,X_n)\colon F\in\mathscr{D}^{(\leqslant)}\} = \sup\left\{\prod_{i=1}^{n} a_i\right\}, \tag{A.14}$$

where $\{a_i\}$ satisfies (A.12) and (A.13).

If $\bar{X}\leqslant 1$ then $F_1\in\mathscr{D}^{(\leqslant)}$ and $\Lambda=1$. Otherwise, let

$$l(\rho)=\log\prod_{i=1}^{n}(1+\rho(X_i-1)). \tag{A.15}$$

Note that l is strictly concave on $[0,1)$ (and $l(\rho)$ is continuous and finite at $\rho=1$ if and only if $\min(X_i)>0$). Further,

$$l'(0)\leqslant 0 \quad \text{if } \bar{X}\leqslant 1. \tag{A.16}$$

Hence, the right-hand side of (4.3) is 1 when $\bar{X}\leqslant 1$, as desired.

By the method of Lagrange multipliers, if $\bar{X}>1$ then the supremum on the right of (A.14) is attained when equality holds in (A.13) and

$$\left(\prod_{j=1}^{n} a_j\right)\Bigg/ a_i=(\lambda_1+\lambda_2 X_i) \quad \forall 1\leqslant i\leqslant n. \tag{A.17}$$

After some manipulation this can be rewritten as

$$a_i=\frac{\lambda}{1+\rho(X_i-1)}, \qquad i=1,\ldots,n, \tag{A.18}$$

where λ, ρ are chosen so that (A.12) and (A.13), with equality, are satisfied. Now consider the two possible cases:

Case 1: $\sum X_i^{-1}>n$. Then we will find a solution to the Lagrange multiplier problem having $a_0=0$. Note that $\sum(X_i-1)a_i=0$ implies

$$l'(\rho)=\sum_{i=1}^{n}\frac{X_i-1}{1+\rho(X_i-1)}=0. \tag{A.19}$$

This equation has a unique positive solution in $(0,1)$ since

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\sum_{i=1}^{n}\frac{X_i-1}{1+\rho(X_i-1)}\right)<0, \quad \sum_{i=1}^{n}(X_i-1)=n(\bar{X}-1)>0$$

$$\text{and}\quad \sum_{i=1}^{n}\frac{X_i-1}{X_i}=n-\sum_{i=1}^{n}X_i<0.$$

Let ρ^* denote this solution. Now note that

$$\sum\frac{1}{1+\rho^*(X_i-1)}=\sum\left(\frac{1}{1+\rho^*(X_i-1)}+\rho^*\frac{X_i-1}{1+\rho^*(X_i-1)}\right)=n. \tag{A.20}$$

Hence, the choice $\rho=\rho^*$ and $\lambda=1/n$ in (A.18) yields the desired supremum. (It is now easy to check that the value $a_0=0$ does in fact correspond to the desired supremum.) Hence (4.3) holds.

Case 2: $\sum X_i^{-1} \leqslant n$. In this case $\min(X_i) > 0$. We find the supremum of $\prod_{i=1}^n a_i$ subject to $a_i \geqslant 0$, and (A.13), with equality. This supremum occurs when

$$\left(\prod_{j=1}^n a_j\right) \Big/ a_i = \lambda_2 X_i \quad \forall 1 \leqslant i \leqslant n. \tag{A.21}$$

Since (A.13) holds with equality we have in this case

$$a_i = \frac{1/n}{X_i} \tag{A.22}$$

and, of course, $a_0 = 1 - n^{-1} \sum_{i=1}^n X_i^{-1} \geqslant 0$. Since $\sum_{i=1}^n a_i \leqslant 1$ in (A.22) it is clear that this solution also yields the supremum subject to (A.12) and (A.13). Hence, $\Lambda = \prod_{i=1}^n X_i$. Furthermore, in this case the supremum on the right of (4.3) is easily seen to be attained when $\rho = 1$, and hence (4.3) again holds, as claimed. □

Proof of Theorem 5.2. The following lemma and its corollary motivate the definition of ϕ^* and are a key part of the proof of the UMP property of ϕ^*. For convenience in the statement of the lemma define for $s \leqslant t$

$$\rho(t,s) = \rho(s,t) = \begin{cases} 1 & \text{if } t \leqslant 1, \\ \frac{1}{s(2t-s)} & \text{if } t > 1, s \geqslant t - \sqrt{t^2 - t}, \\ 1 - \frac{(t-1)^2}{(t-s)^2} & \text{if } t > 1, s < t - \sqrt{t^2 - t}. \end{cases} \tag{A.23}$$

An algebraic calculation shows that ϕ^* as defined by (5.3) satisfies

$$\phi^*(s,t) = \begin{cases} 1 & \text{if } \rho(s,t) \leqslant \alpha, \\ 0 & \text{if } \rho(s,t) > \alpha. \end{cases} \tag{A.24}$$

Lemma A.1. *For any* $0 \leqslant s \leqslant t$ *with* $t > 1$ *let* $\mathscr{P}(s,t)$ *denote the collection of probability distributions supported on the set* $\{0, s, t\}$ *and having mean* $\leqslant 1$. *Then*

$$\sup_{P \in \mathscr{P}(s,t)} P(X_{(1)} \geqslant s, X_{(2)} \geqslant t) = \rho(s,t). \tag{A.25}$$

This supremum occurs for the distribution $P_{s,t}^* = P^*$ *given by* (A.26)–(A.30) *below*:
When

$$0 \leqslant s < t - \sqrt{t^2 - t} \quad (\Rightarrow s < 1) \tag{A.26}$$

then

$$P^*(s) = \frac{t-1}{t-s}, \quad P^*(t) = \frac{1-s}{t-s}, \quad \textit{and} \quad P^*(0) = 0 \ \textit{if} \ s \neq 0. \tag{A.27}$$

When

$$t \geqslant s \geqslant t - \sqrt{t^2 - t} \left(\Rightarrow s > \frac{1}{2}\right) \tag{A.28}$$

then

$$P^*(s) = \frac{t-s}{s(2t-s)}, \qquad P^*(t) = \frac{1}{2t-s}, \tag{A.29}$$

$$P^*(0) = \frac{s(t-s) - t(1-s)}{s(2t-s)} = 1 - \frac{t}{s(2t-s)} \geqslant 0. \tag{A.30}$$

(*Note that* (A.28) *entails* $s(2t-s) \geqslant t$.)

Proof. For any distribution $P \in \mathscr{P}(s,t)$ let $p_s = P(\{s\})$, $p_t = P(\{t\})$. Then $p_s, p_t \geqslant 0$

$$p_s + p_t \leqslant 1 \tag{A.31}$$

and

$$s p_s + t p_t \leqslant 1. \tag{A.32}$$

Conversely, if $p_s, p_t \geqslant 0$ satisfying (A.31) and (A.32) then there is a corresponding $P \in \mathscr{P}(s,t)$ having $P(0) = 1 - p_s - p_t$. Hence, it suffices to consider the problem of maximizing

$$P(X_{(1)} \geqslant s,\ X_{(2)} \geqslant t) = 2 p_s p_t + p_t^2 \tag{A.33}$$

subject to (A.31), (A.32) and $p_s, p_t \geqslant 0$.

When $t > 1$ the maximum can occur either when both (A.31) and (A.32) are equalities or when only (A.32) is an equality. (It is easy to check that when $t > 1$ the maximum cannot occur when (A.32) is an inequality.) The first case directly yields (A.27) to get $p_s = (1 - t p_t)/s$, substitute this into the objective function $2 p_s p_t + p_t^2$, and then maximize over p_t. The values at which the maximum occurs are given by (A.29) and these satisfy (A.31) whenever $P^*(0)$ given by (A.30) satisfies $p^*(0) > 0$. Finally, if $p^*(0) > 0$ substitute (A.27) into the objective function and (A.29) into that function and compare the resulting values. The larger of these two values is ρ as given by (A.23). (Conditions (A.26) and (A.28) describe the regions where (A.27) and (A.29), respectively, maximize the objective function.) □

This is a standard problem of maximization under inequality constraints. The maximum occurs when (A.31) and (A.32) are satisfied and

$$2 p_s + 2 p_t = \lambda_1 + t\lambda_2, \tag{A.34}$$

$$2 p_t = \lambda_1 + s\lambda_2. \tag{A.35}$$

Here, $\lambda_2 = 2 p_s/(t-s) > 0$ and equality holds in (A.32). Also $\lambda_1 \neq 0$ only if $p_s + p_t = 1$. In that case $\lambda_1 = 2(s^2 - 2st + t)/(t-s)^2 \geqslant 0$. Overall, the solution to these Lagrange multiplier equations is given by (A.26)–(A.30), and yields (A.25) as the maximizing value of $R(p_s, p_t)$. □

Corollary A.1. *If ϕ is a strongly monotone level-α test of* H_0 *versus* H_a *then*

$$\phi \leqslant \phi^*. \tag{A.36}$$

Proof. Suppose (A.36) does not hold. Then there is some $0 \leqslant s' \leqslant t'$ such that

$$\phi(s',t') > 0, \qquad \phi^*(s',t') = 0. \tag{A.37}$$

It follows from this, strong monotonicity, and the upper semicontinuity of ϕ^* that there is some $0 \leqslant s \leqslant t$ for which

$$\phi(s,t) = 1, \qquad \phi^*(s,t) = 0. \tag{A.38}$$

Note that (A.38) also implies $1 = \phi(t,s) = \phi(t,t)$. It then follows from the lemma and property (A.24) of ϕ^* that

$$\sup_{P \in \mathscr{P}(s,t)} E_P(\phi(X_1,X_2)) \geqslant \rho(s,t) > \alpha. \tag{A.39}$$

Hence ϕ is not of level α. □

The following two technical lemmas are used in the proof of the main theorem. For $0 \leqslant a_0 \leqslant \cdots \leqslant a_k$ let

$$\mathscr{P}(a_0,\ldots,a_k) = \{P \in \mathscr{P}_0\text{: } P \text{ supported on } \{a_0,\ldots,a_k\}\}. \tag{A.40}$$

For P supported on $\{a_0,\ldots,a_k\}$ let $p_i = P(\{a_i\})$, $i = 0,\ldots,k$, so that $E_P(X) = \sum_{i=0}^k a_i p_i$.

Lemma A.2. *Assume $1/\sqrt{\alpha} \leqslant a_k \leqslant [1+\sqrt{(1-\alpha)}]/\alpha = \hat{T}$, Let $a_0 = 0, a_1 = s(a_k)$. Then*

$$\sup_{P \in \mathscr{P}(a_0,\ldots,a_k)} \left\{ P^*(a_k) \sum_{i=1}^k p_i + P^*(a_1) p_k \right\} = \alpha, \tag{A.41}$$

and the supremum is uniquely obtained when $p_1 = P^*(a_1)$, $p_k = P^*(a_k)$, $p(0) = P^*(0)$ with $P^* = P^*_{a_1,a_k}$ as in (A.26)–(A.29).

Proof. As in the proof of Lemma A.1, the maximizing $P \in \mathscr{P}(a_0,\ldots,a_k)$ must satisfy

$$P^*(a_1) + P^*(a_k) = \lambda_1 + \lambda_2 a_k, \tag{A.42}$$

and

$$P^*(a_k) = \lambda_1 + \lambda_2 a_j \tag{A.43}$$

for every $j = 1,\ldots,k-1$ such that $p_j \neq 0$. Also, $p_j = 0$, $j = 1,\ldots,k-1$ only for those indices, j, for which the left-hand side of (A.43) is smaller than the right-hand side. (Also, $\lambda_1 = 0$ unless $p_0 = 0$.) It follows that the unique solution to (A.42), (A.43) maximizing (A.41) has λ_1, λ_2 as following (A.34), (A.35), with $s = a_1$, $t = a_k$, and $P = P^*_{s,t}$. □

Lemma A.3. *Both $0 < s(t) < 1$ and $P^*_{s,t}(0) > 0$ if and only if*

$$\frac{1+\alpha^{-1}}{2} < t < \alpha^{-1}. \tag{A.44}$$

Let t satisfy (A.44) *and* $s = s(t)$. *Let* $P \in \mathscr{P}_0(s,t)$. *Then*

$$\begin{aligned}(P^*(t) - P(t))P(s) - P^*(0)P(t) &= \left(\frac{1}{2t-s} - \frac{1-s}{t-s}\right)\left(\frac{t-1}{t-s}\right)\\ &\quad - \left(\frac{2st-s^2-t}{s(2t-s)}\right)\left(\frac{1-s}{t-s}\right)\\ &> 0. \end{aligned} \tag{A.45}$$

Proof. From (5.2), $s(1+\alpha^{-1}/2) = 1$. Hence, $s(t) < 1$ if and only if $t > (1+\alpha^{-1})/2$. Also $P_{s,t}(0) > 0$ only when $t < 1/\alpha$, This verifies the assertion at (A.44).

$\mathscr{P}_0(s,t) \neq \phi$ since $s < 1$. In fact, then $\mathscr{P}_0(s,t)$ contains only the distribution, P, having

$$p_1 = P(s) = \frac{t-1}{t-s}, \quad p_2 = \frac{1-s}{t-s}. \tag{A.46}$$

It is then possible to directly verify the inequality in (A.45), however, the following indirect proof may be more informative.

Let $Q_\epsilon = (1-\epsilon)P + \epsilon P^*_{s,t} \in \mathscr{P}_0(0,s,t)$. It can be checked directly from (5.2) that

$$p_2 < P^*(t), \quad p_1 > P^*(s). \tag{A.47}$$

Hence, Q_ϵ is a distribution whenever

$$0 \leqslant \epsilon \leqslant p_1/(p_1 - P^*(s)). \tag{A.48}$$

Now by Lemma A.1

$$g(\epsilon) = 2Q_\epsilon(s)Q_\epsilon(t) + Q_\epsilon^2(t) \tag{A.49}$$

is uniquely maximized at $\epsilon = 1$. Furthermore, $g(\epsilon)$ is a quadratic in ϵ, and $\epsilon = 1$ is in the interior of its range of definition (since $p_1/(p_1 - P^*(s)) > 1$). Hence, $g'(0) > 0$. Thus,

$$\begin{aligned}0 < g'(0) &= 2p_2(P^*(s) - p_1) + 2p_1(p^*(t) - p_2) + 2p_2(P^*(t) - p_2)\\ &= 2\{p_2(P^*(s) + P^*(t) - p_1 - p_2) + p_1(P^*(t) - p_2)\}\\ &= 2\{-p_2P^*(0) + p_1(P^*(t) - p_2)\}. \end{aligned} \tag{A.50}$$

This verifies (A.45). $\square$

Proof of the Theorem. In view of Corollary A.1 it remains only to show that ϕ^* has level α. So, fix $\alpha, 0 < \alpha < 1$. For this purpose it then suffices to show that for any finite set $A = \{a_0, \ldots, a_k\}$

$$\sup_{P \in \mathscr{P}_0(A)} E_P(\phi^*(X_1, X_2)) \leqslant \alpha. \tag{A.51}$$

There is no loss of generality in assuming that $1/\sqrt{\alpha} \in A$ (and hence $a_k \geqslant 1/\sqrt{\alpha}$), $\hat{T} \in A$, and that for $0 < a < 1/\sqrt{\alpha}$, $a \in A$ if and only if $s^{-1}(a) \in A$. For any fixed set, A, of this form the supremum in (A.51) is attained. Let $\tilde{P} \in \mathscr{P}_0(A)$ be a distribution which attains this supremum. We will show that $\tilde{P} = P^*_{s,t}$ for some t and $s = s_\alpha(t)$.

We can write $A = \{a_0, a_1, \ldots, a_l, 1/\sqrt{\alpha}, a_{m-l}, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, a_k\}$ where $a_0 = 0$, $a_{l+1} = 1/\sqrt{\alpha}$, $a_m = \hat{T}$, $a_i = s(a_{m-i})$ for $i = 1, \ldots, l$, and $a_i > \hat{T}$ for $i = m+1, \ldots, k$. (Note that $m = 2l + 2$ by virtue of the definition of A.)

Let

$$\lambda = \min\{i\colon\ i \geqslant 1,\ \tilde{p}_i > 0\},\ \xi = \max\{i\colon\ \tilde{p}_i > 0\}. \tag{A.52}$$

We now proceed to show

$$\xi \leqslant m, \tag{A.53}$$

$$\xi = m \Rightarrow \tilde{p}_0 > 0, \tag{A.54}$$

$$l + 1 \leqslant \xi < m \Leftrightarrow \lambda = m - \xi. \tag{A.55}$$

(Hence, $\xi = l + 1 \Leftrightarrow \lambda = l + 1$.)

Suppose first that $\xi > m$, or $\xi = m$ and $\tilde{p}_0 = 0$, or $\xi \geqslant l + 2$ and $0 < m - \xi < \lambda$. Then for some $\epsilon > 0$ define a new distribution $P \in \mathscr{P}_0(A)$ by

$$
\begin{aligned}
p_\xi &= \tilde{p}_\xi - \epsilon,\\
p_{\xi-1} &= \tilde{p}_{\xi-1} + \epsilon \frac{a_\xi - a_{\tilde{\lambda}}}{a_{\xi-1} - a_{\tilde{\lambda}}},\\
p_{\tilde{\lambda}} &= \tilde{p}_{\tilde{\lambda}} - \epsilon \frac{a_\xi - a_{\xi-1}}{a_{\xi-1} - a_{\tilde{\lambda}}},
\end{aligned} \tag{A.56}
$$

where $\tilde{\lambda} = 0$ if $\tilde{p}_0 > 0$ and $\tilde{\lambda} = \lambda$ otherwise. Also, let $p_i = \tilde{p}_i$ for $i \neq \tilde{\lambda}, \xi - 1, \xi$.

Since both $\tilde{p}_\xi > 0$ and $p_{\tilde{\lambda}} > 0$, one can choose $\epsilon > 0$ so that all $p_i \geqslant 0$. It is then easy to check that $P \in \mathscr{P}_0(A)$. Furthermore, $\phi^*(a_\xi, a_i) = \phi^*(a_{\xi-1}, a_i)$ for all i having $\tilde{p}_i > 0$ and $\phi^*(a_{\tilde{\lambda}}, a_i) \leqslant \phi^*(a_{\xi-1}, a_i)$ with strict inequality for $i = \tilde{\lambda}$. It follows that

$$E_p(\phi^*(X_1, X_2)) > E_{\tilde{p}}(\phi^*(X_1, X_2)). \tag{A.57}$$

This contradicts the assumption that $\tilde{P}$ maximizes (A.51). Hence, $\xi \leqslant m$ and $\tilde{p}_{m-\xi} > 0$.

Similarly, if $\tilde{p}_i = 0$ for $i \geqslant m$ and $\tilde{p}_{m-\lambda} = 0$ then define

$$
\begin{aligned}
p_\lambda &= 0,\\
p_{\lambda-1} &= \tilde{p}_{\lambda-1} + \tilde{p}_\lambda \frac{a_m - a_\lambda}{a_m - a_{\lambda-1}},\\
p_m &= \tilde{p}_\lambda \frac{a_\lambda - a_{\lambda-1}}{a_m - a_{\lambda-1}}.
\end{aligned} \tag{A.58}
$$

Then, $P \in \mathscr{P}_0(A)$ and (A.57) holds since $\phi^*(a_{\lambda-1}, i) = \phi^*(a_\lambda, i)$ for all i having $\tilde{p}_i > 0$, and $\phi^*(a_{\lambda-1}, a_m) = 1$ with $p_m > 0$ but $\tilde{p}_m = 0$. Again, this contradicts the assumption that $\tilde{p}$ maximizes (A.51).

The assertions following (A.56) and (A.58) combine to yield (A.53)–(A.55).

Now assume $\tilde{p}_0 > 0$ and either $\xi = m$ or $l \geqslant \lambda = m - \xi \geqslant 1$. Let $P^* = P_{a^*_\lambda, a_\xi}$ and

$$P_\epsilon = (1-\epsilon)\tilde{P} + \epsilon P^*,$$
$$g(\epsilon) = E_{P_\epsilon}(\phi^*(X_1, X_2)). \tag{A.59}$$

Suppose (A.51) is false — i.e., suppose $E_{\tilde{P}}(\phi^*(X_1,X_2)) > \alpha$, Then

$$g'(0) = -2\alpha + 2\left\{P^*(a_\xi)\sum_{i=\lambda}^{\xi}\tilde{p}_i + P^*(a_\lambda)\tilde{p}_\xi\right\} < 0 \tag{A.60}$$

by Lemma A.2 unless $\tilde{P} = P^*$. Since there exist values $\epsilon < 0$ such that $P_\epsilon \in \mathscr{P}_0(A)$ it follows that either $\tilde{P} = P^*$ and hence (A.51) holds, or $\tilde{P}$ does not yield the supremum in (A.51), a contradiction.

If $\tilde{p}_0 > 0$ and $\lambda = l+1 = \xi$ then $\tilde{P}$ is concentrated on only the two points $a_0, a_{l+1} = 1/\sqrt{\alpha}$. In this case P^* is also concentrated on just those two points. Hence, again (A.51) holds.

Finally, if $\tilde{p}_0 = 0$ and $l \geqslant \lambda = m - \xi \geqslant 1$ then $a_\lambda = \min\{supp\,\tilde{P}\}$. Hence, $a_\lambda < 1$ since $a_\xi > 1$, $\tilde{p}_\xi > 0$ and $E_{\tilde{P}}(X) = 1$.

There are two possibilities, either $P^*(0) = 0$ or $P^*(0) > 0$. In the former case the reasoning involving (A.60) yields that (A.51) holds. If, instead, $P^*(0) > 0$ then we apply Lemma A.3. First define $Q \in \mathscr{P}_0$ by

$$Q(\{a_\xi\}) = 1 - Q(\{a_\lambda\}) = \frac{1 - a_\lambda}{a_\xi - a_\lambda}. \tag{A.61}$$

Then let

$$\check{P}_\epsilon = \tilde{P} + \epsilon(P^* - Q),$$
$$\check{g}(\epsilon) = E_{\check{P}_\epsilon}(\phi^*(X_1, X_2)). \tag{A.62}$$

Note that $\check{P}_\epsilon \in \mathscr{P}_0$ for $\epsilon \geqslant 0$ sufficiently small and that $\tilde{p}_\xi \leqslant q_\xi$ since $E_{\tilde{P}}(X) = 1 = E_Q(X)$ and $supp\,\tilde{P} \subset [a_\lambda, a_\xi]$. Then

$$\begin{aligned}
\check{g}'(0^+) &= 2\{(p^*_\xi - q_\xi) + (p^*_\lambda - q_\lambda)\tilde{p}_\xi\} \\
&\geqslant 2\{(p^*_\xi - q_\xi)(q_\lambda + q_\xi) + (p^*_\lambda - q_\lambda)q_\xi\} \\
&\quad (\text{since } \tilde{p}_\xi \leqslant q_\xi \text{ and } p^*_\lambda - q_\lambda < 0, \text{ and } q_\xi + q_\lambda = 1) \\
&= 2\{(p_\xi - q_\xi)q_\lambda - p^*_0 q_\xi\} > 0,
\end{aligned} \tag{A.63}$$

by Lemma A.3. It follows that $\tilde{P}$ is again not the supremum in (A.51), a contradiction.

It follows from the above arguments that $\tilde{P}$ satisfies (A.53)–(A.55). It also follows that $\tilde{P} = P^*$, for otherwise strict inequality holds in (A.60) or (A.63) (whichever is appropriate). But then (A.51) holds since $E_{P^*_{a_\xi}}(\phi^*(X_1, X_2)) = \alpha$. This completes the proof of the theorem. $\square$

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