1. In an urn there are two balls, one with the number 2, and one with the number 3. When you pick from the urn the chance of observing a ball with the number 2 on it is $\frac{1}{2}$, and the chance of observing the number 3 is also $\frac{1}{2}$. Now pick with replacement from the urn until the sum of the scores is 8 or more. Let $N$ be the number of balls chosen and $M$ be the sum of the scores.

(a) Write down the joint probability mass function for $N$ and $M$.
(b) Find the marginal probability mass function for $M$.
(c) Find $P(N = 4 | M = 9)$.
(d) Find $E(M - 2N)$.

Solution

(a) • $\{3, 3, 3\}$
   $P(N = 3, M = 9) = (\frac{1}{2})^3$

   • $\{3, 3, 2\} \times 3$
   $P(N = 3, M = 8) = (\frac{1}{2})^3 \times 3$

   • $\{3, 2, 2, 2\} \times 4$
   $P(N = 4, M = 9) = (\frac{1}{2})^4 \times 4$

   • $\{2, 2, 2, 2\}$
   $P(N = 4, M = 8) = (\frac{1}{2})^4$

   • $\{2, 2, 3, 3\} + \{2, 3, 2, 3\} + \{3, 2, 2, 3\}$
   $P(N = 4, M = 10) = (\frac{1}{2})^4 \times 3$

(b) $P(M = 8) = P(N = 3, M = 8) + P(N = 4, M = 8) = (\frac{1}{2})^3 \times 3 + (\frac{1}{2})^4 = \frac{7}{16}$

   $P(M = 9) = P(N = 3, M = 9) + P(N = 4, M = 9) = (\frac{1}{2})^3 + (\frac{1}{2})^4 \times 4 = \frac{3}{8}$

   $P(M = 10) = P(N = 4, M = 10) = (\frac{1}{2})^4 \times 3 = \frac{3}{16}$

(c) $P(N = 4 | M = 9) = \frac{P(M = 9, N = 4)}{P(M = 9)} = \frac{(\frac{1}{2})^4 \times 4}{(\frac{1}{2})^3 + (\frac{1}{2})^4 \times 4} = \frac{2}{3}$

(d) $E(M - 2N) = E(M) - 2E(N)$

   $= 8 \cdot P(M = 8) + 9 \cdot P(M = 9) + 10 \cdot P(M = 10) - 2[3 \cdot P(N = 3) + 4 \cdot P(N = 4)]$

   $= 8 \cdot \frac{7}{16} + 9 \cdot \frac{3}{8} + 10 \cdot \frac{3}{16} - 2[3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2}]$

   $= \frac{7}{4}$
• \( P(N = 3) = \left(\frac{1}{2}\right)^3 + 3 \left(\frac{1}{2}\right)^3 = \frac{1}{2} \)

• \( P(N = 4) = 4 \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 \times 3 = \frac{1}{2} \)

2. We saw in class that a coin-flipping game that pays \$2^n if the first head appears on the \(n^{th}\) (that is, a random variable \(X\) such that \(P(X = 2^n) = 1/2^n\) for \(n = 1, \ldots, \infty\)) toss has infinite expectation.

(a) Economists, among others, like to speak in terms of utility functions. Intuitively, you certainly would value an additional dollar more if you had fewer to begin with. Let \(U(x) = x^{1/k}\) for \(k > 0\). Compute \(E(U(X))\) and note for which values of \(k\) this converges. What property does \(U\) have for these values of \(k\)?

(b) If \(k = 2\), then \(U(X)\) has finite expectation. Compute the variance. Notice anything strange?

(c) Now let \(U(x) = \log_2(x)\). Repeat the first part with this function. Is this value larger or smaller. (Log-utility is very common in economic theory.)

Solution

(a)

\[
E(U(X)) = \sum_{n=1}^{\infty} \frac{2^{n/k}}{2^n} = \sum_{n=1}^{\infty} 2^{(1-k)n/k}
\]

Notice that for \(k > 1\) this is a geometric series so the sum is \(\frac{2^{(1-k)/k}}{1-2^{(1-k)/k}} = \frac{1}{2^{(1-k)/k}-1}\). For \(k > 1\) \(U(X)\) is concave. Notice also that as \(k \to 1\) the expected value goes to \(\infty\).

(b)

\[
E((U(X))^2) = \sum_{n=1}^{\infty} \left(\frac{2^{n/k}}{2^n}\right)^2 = \sum_{n=1}^{\infty} 2^{(2-k)n/k}
\]

When \(k = 2\) this is infinite. So even though the mean is finite for \(k = 2\), the variance is still infinite.

(c)

\[
E(\log_2 X) = \sum_{n=1}^{\infty} \frac{n}{2^n}
\]

Call this sum \(s\). Notice that \(s - \frac{1}{2} s = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1\). Solving for \(s\), this gives \(E(\log_2 X) = 2\). In the first part, the \(k\) that gives the same expected utility is \(\frac{\log(2)}{2\log(2)-\log(3)} \approx 2.41\)

3. Let \(N\) be the number of times you need to throw a six-sided until you have observed each of the sides. For example suppose you start throwing the die and you see 3,4,3,5,1,1,6,2. You would then stop and \(N\) would then be 8.

(a) Find \(E(N)\).

(b) Find \(Var(N)\).

Solution Let \(X_i\) be the number of times we have to roll the die until we see the \(i^{th}\) unique number. Note that each of these are independent and that \(X_i \sim Geom((6-i+1)/6)\). Now, \(N = X_1 + X_2 + \cdots + X_6\)

(a) \(E(N) = \sum_{i=1}^{6} \frac{1}{(6-i+1)/6} = 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7\)
(b) We know that variance of a geometric random variable with parameter \( p \) is \( \text{Var}(X) = \frac{1-p}{p^2} \).

By independence, \( \text{Var}(N) = \sum_{i=1}^{6} \text{Var}(X_i) = \sum_{i=1}^{6} \frac{(i-1)/6}{(6-i+1)/6} = 38.99 \)

4. Let \( X \) and \( Y \) be independent random variables. Suppose that \( E(X) = 1, \ E(Y) = 0, \ \text{Var}(X) = 10, \ \text{Var}(Y) = 9. \) Find \( E(X - 2Y)^2 \).

Solution \( E(X - 2Y)^2 = E(X^2 - 4XY + 4Y^2) = E(X^2) - 4E(XY) + 4E(Y^2) = 11 + 36 = 47 \)

\( E(X^2) = \text{Var}(X) + (E(X))^2 = 10 + 1 = 11 \)

\( E(Y^2) = \text{Var}(Y) + (E(Y))^2 = 9 + 0 = 9 \)

\( E(XY) = E(X)E(Y) = 0 \)

5. A particular binary data transmission and reception device is prone to some error when receiving data. Suppose that each bit is read correctly with probability \( p \).

(a) Find a value of \( p \) such that when 10,000 bits are received, the expected number of errors is at most 10.

(b) Using this value of \( p \) what is the probability of no errors? Of at least 1? Of at least 10?

Solution This a binomial random variable with \( n = 10,000 \) and \( p \). Then the expectation \( E(X) = np \)

(a) \( E(X) = np \geq 10,000 - 10 = 9900 \)

\( p = 0.999 \)

(b) \( P(\text{no errors}) = p^n = 0.999^{10000} = 0.0000452 \)

\( P(\text{at least one}) = 1 - P(\text{no errors}) = 1 - 0.0000452 = 0.9999548 \)

\( P(\text{at least 10}) = 1 - P(\text{no errors}) - P(1 \text{ error}) - P(2 \text{ errors}) - \cdots - P(9 \text{ errors}) \)

\( = 1 - p^n - \sum_{i=1}^{9} \binom{n}{i} (1-p)^i p^{n-i} \)

Alternatively, since \( n \) is large and the probability of an error is small, we could approximate the number of errors as \( \text{Pois}(np) = \text{Pois}(10) \).

6. Let \( X \) be a discrete random variable that is uniformly distributed over the set of integers in the range \([a, b]\), where \( a \) and \( b \) are integers with \( a < 0 < b \). Find the PMF of the random variables \( \max\{0, X\} \) and \( \min\{0, X\} \)

Solution Notice that \( \max\{0, X\} = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X \leq 0 \end{cases} \). So, \( P(\max\{0, X\} = 0) = P(X \leq 0) = \frac{|a|+1}{b-a+1}. \) For \( k > 0 \) we just have \( P(\max\{0, X\} = k) = \frac{1}{b-a+1} \). Similarly, for the minimum, we have \( P(\min\{0, X\} = 0) = P(X \geq 0) = \frac{b+1}{b-a+1} \) and \( P(\max\{0, X\} = k) = P(X = k) \) for \( k < 0 \).