Some Counterclaims Undermine Themselves in Observational Studies

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- If I am prosecuting A for murder, I might observe that the victim encountered only A, B and C on the day of his death, and it is not plausible that B or C murdered the victim.
- Proof by contradiction: I argue for $\mathcal{T}$ by showing that $\sim \mathcal{T}$ leads to a contradiction. The supposition that $\sim \mathcal{T}$ undermines itself.

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- A claim $\mathcal{T}$ and counterclaims to $\mathcal{T}$ may be offered by different people, say an investigator and a critic.
- Or an investigator may anticipate certain counterclaims to $\mathcal{T}$ and try to strengthen the case for $\mathcal{T}$ by refuting or rendering implausible various counterclaims to $\mathcal{T}$.

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- Could empirical evaluation of such a counterclaim show that it fails as a counterclaim? That it does not make the original claim less plausible.

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- In this sense, the counterclaim undermines itself. It fails in its role as a counterclaim.

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- Is what we saw in the example expected under simple models for treatment effects? (Design sensitivity and power of a sensitivity analysis.)

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- The preliminary fact: adjusting for an outcome can bias an otherwise unbiased estimate of a treatment effect.

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- If you adjusted for posttreatment blood pressure, then you might remove the genuine effect of the antihypertensive drug.
- If the drug worked by lowering your blood pressure so that you had the same low risk of stroke as a person with naturally low blood pressure, that might be a large effect, and you might mistakenly remove it.

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- The system has little information about events leading up to the crash: speeds, distances between vehicles, road traction, driver performance, condition of brakes, etc, all of which affect the forces involved in the crash.

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- What can be done?

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  3. People in Volvos and Mercedes are more likely to be belted than people in Fords.
- People aged 18–30 are twice as likely as older individuals to be unbelted (odds ratio 2.1). Unbelted individuals were on average 9 years younger than belted individuals.

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- In Evan's comparison, many unmeasured factors are controlled: same vehicle in same crash, driver and passenger traveled at the same speed, at the same distance from the car ahead, with the same road traction.
- The risks in the driver's seat may differ from those in the passenger's seat, but we see both cases.

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- Notation will describe any one of the 4 studies, so the notation is recycled.

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- $Y_i =$ driver - minus - passenger difference in injury scores, -4 to 4. So a -4 means the driver was not injured but the passenger died.

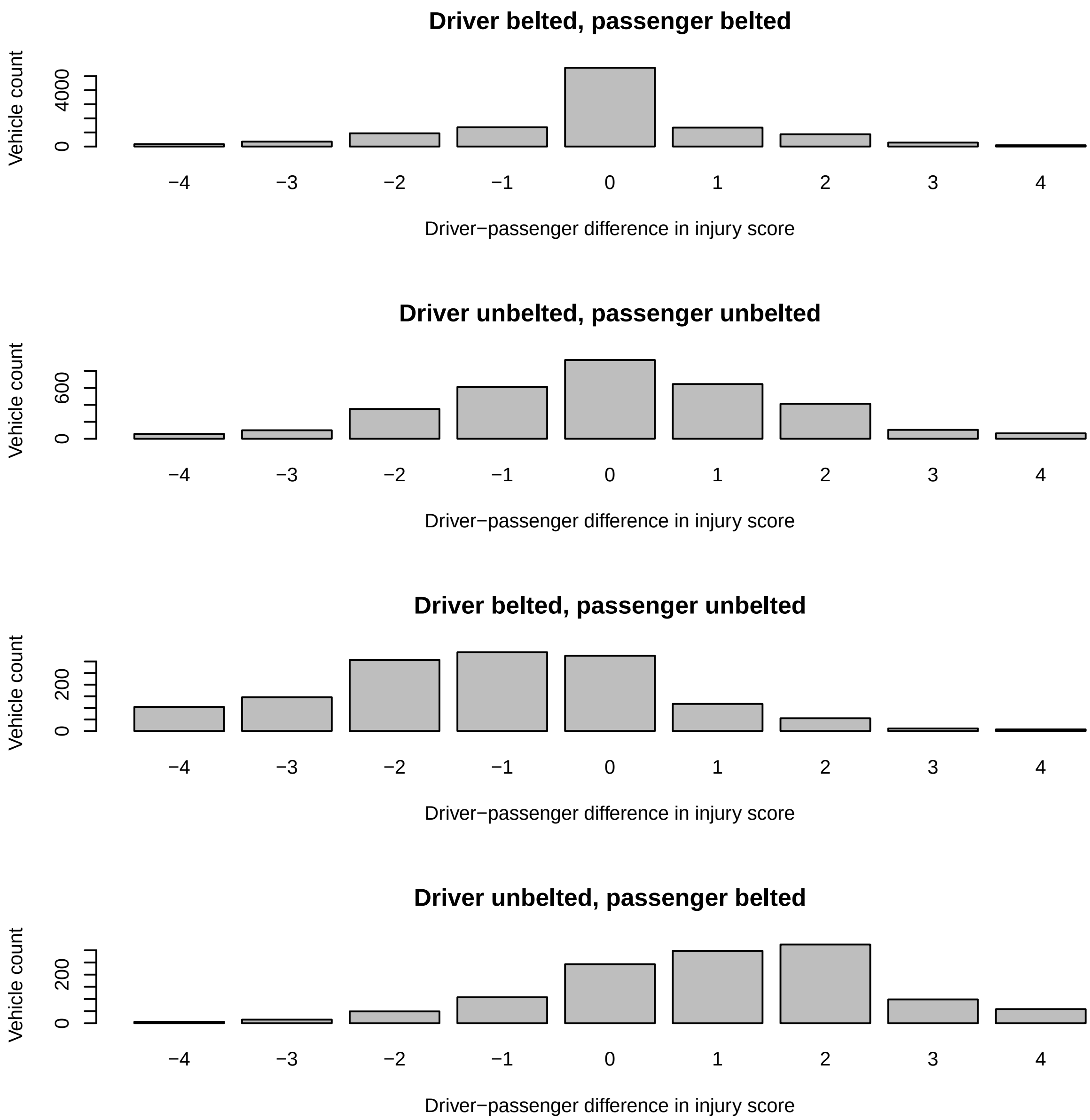

Figure 1: Pair differences in injury scores, driver-minus-passenger, for a driver and a passenger in the same car in FARS 2010-2011, by restraint use. A positive difference indicates the driver suffered more severe injuries than the passenger.

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  4. (n, ls) is 1.34 years.

Notation for any one of our 4 studies (e.g., (ls, n), etc.)

- I matched sets, $i \in \{1, \ldots, I\} = \mathcal{I}$, where set $i \in \mathcal{I}$ contains subjects $\mathcal{J}_i = \{1, \ldots, J_i\}$, so ij is a person. (In the example, $J_i = 2$ and $\mathcal{J}_i = \{1, 2\}$ for all i.)

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- Let $\mathcal{Z}$ be the set containing the $\prod_{i \in \mathcal{I}} J_i$ possible values of $\mathbf{Z}$, so $\mathbf{z} \in \mathcal{Z}$ if $\mathbf{z}$ is of dimension n with $z_{ij} = 0$ or $z_{ij} = 1$ and $1 = \sum_{j \in \mathcal{J}_i} z_{ij}$ for each i. Conditioning on $\mathbf{Z} \in \mathcal{Z}$ is abbreviated as conditioning on $\mathcal{Z}$.

Notation for any one of our 4 studies (e.g., (ls, n), etc.)

- I matched sets, $i \in \{1, \ldots, I\} = \mathcal{I}$, where set $i \in \mathcal{I}$ contains subjects $\mathcal{J}_i = \{1, \ldots, J_i\}$, so ij is a person. (In the example, $J_i = 2$ and $\mathcal{J}_i = \{1, 2\}$ for all i.)
- Set i contains one treated with $Z_{ij} = 1$, the rest untreated controls with $Z_{ij} = 0$, so $1 = \sum_{j \in \mathcal{J}_i} Z_{ij}$ for each i.
- Write $\mathbf{Z} = \left(Z_{11}, Z_{12}, \ldots Z_{IJ_I}\right)^T$ for the vector of dimension $n = \sum_{i \in \mathcal{I}} J_i$
- Let $\mathcal{Z}$ be the set containing the $\prod_{i \in \mathcal{I}} J_i$ possible values of $\mathbf{Z}$, so $\mathbf{z} \in \mathcal{Z}$ if $\mathbf{z}$ is of dimension n with $z_{ij} = 0$ or $z_{ij} = 1$ and $1 = \sum_{j \in \mathcal{J}_i} z_{ij}$ for each i. Conditioning on $\mathbf{Z} \in \mathcal{Z}$ is abbreviated as conditioning on $\mathcal{Z}$.
- Denote by $|\mathcal{A}|$ the number of elements in a finite set $\mathcal{A}$ so that, for instance, $|\mathcal{J}_i| = J_i$ and $|\mathcal{Z}| = \prod_{i \in \mathcal{I}} J_i$.

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- In the example, $\mathbf{x}_i$ describes the vehicle and the crash.
- Quite possibly $u_{ij} \neq u_{ik}$ for many i, j, k.
- Example: u_{ij} is a measure of the frailty of individual ij, and there is concern that frail individuals are less likely to wear safety belts and more likely to be suffer severe injuries or death.

Outcomes (in each of our 4 parallel studies, e.g., (ls, n).)

- Subject ij has two potential injury scores, r_{Tij} if assigned to treatment or r_{Cij} if assigned to control, so the observed response of ij is $R_{ij} = Z_{ij}\, r_{Tij} + (1 - Z_{ij})\; r_{Cij}$, and the effect of the treatment on ij, namely $r_{Tij} - r_{Cij}$ is not observed; see Neyman (1923) and Rubin (1974).

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- Treated-minus-control pair difference $Y_i = (Z_{i1} - Z_{i2})\,(R_{i1} - R_{i2})$ in outcomes.

Notation, continued.

- Write $\mathcal{F} = \left\{ \left(r_{Tij}, r_{Cij}, \mathbf{s}_{Tij}, \mathbf{s}_{Cij}, \mathbf{x}_{ij}, u_{ij} \right),\ i = 1, \ldots, I,\ j = 1, \ldots, J_i \right\}$.

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- The subscripts ij are unique but noninformative identifiers, perhaps randomly assigned, and all information about individual ij is in observed or unobserved variables that describe ij.

Randomization inference (in each of our 4 parallel studies, e.g., (ls, n).)

- If this were a randomized experiment, then we would, independently, assign treatment at random to one person in each matched set, so

$$\Pr\left(\mathbf{Z} = \mathbf{z} \mid \mathcal{F}, \mathcal{Z}\right) = \prod_{i \in \mathcal{I}} J_i^{-1} = |\mathcal{Z}|^{-1} \text{ for each } \mathbf{z} \in \mathcal{Z}.$$

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3. $\mathbf{Z}$ is uniform on $\mathcal{Z}$ in a randomized experiment.

Test statistics

- Huber-Maritz M-tests $t(\mathbf{Z}, \mathbf{R}) = \sum_{i=1}^{I} \psi(Y_i / s)$ where s is the 95% quantile of $|Y_i|$, and $\psi(\cdot)$ is an odd function, $\psi(y) = -\psi(-y)$.

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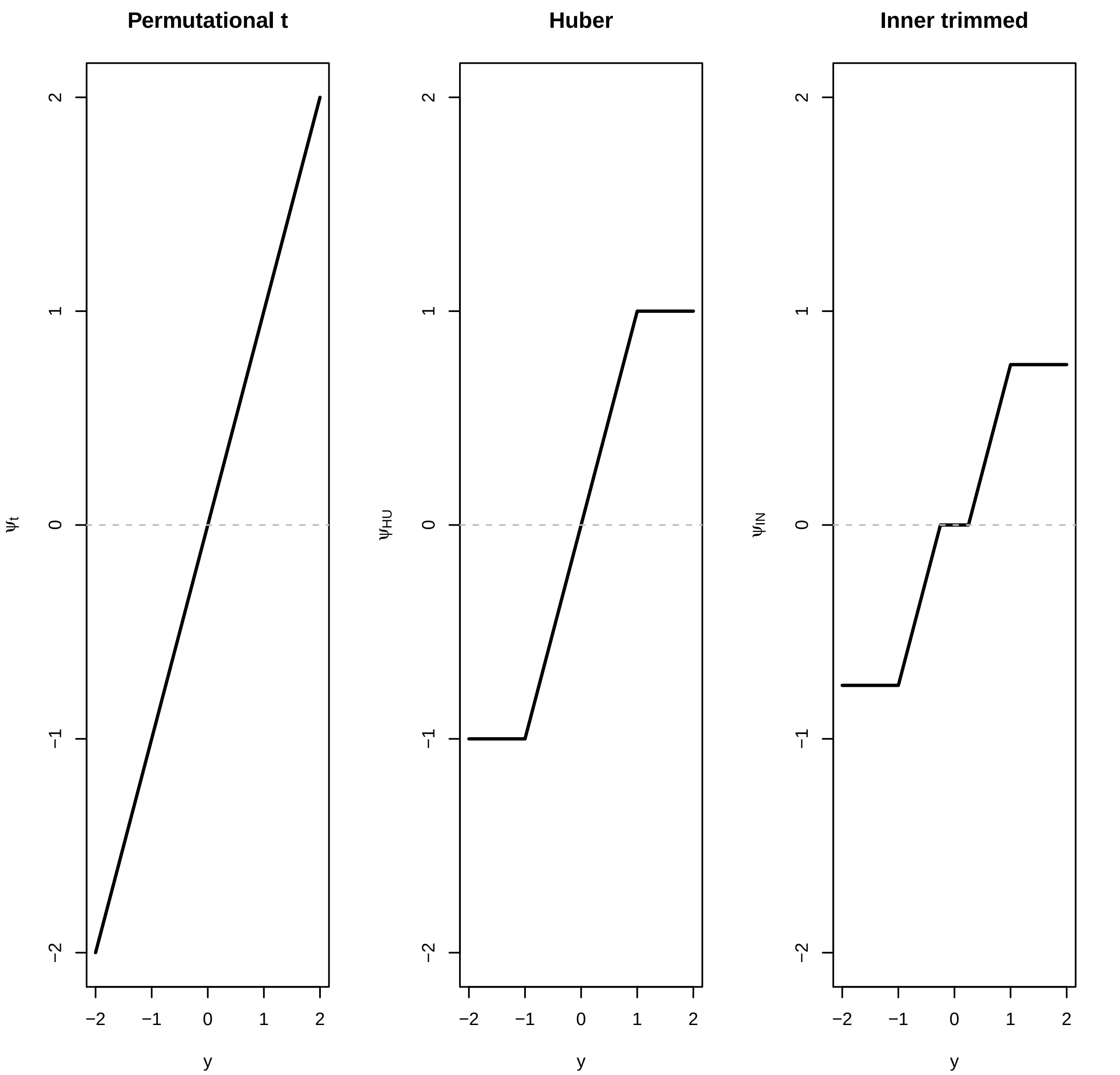
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- I.e., the null distribution of $\sum_{i=1}^{I} \psi(Y_i / s)$ has a simple form.

Descriptive statistics and randomization tests

Table: Randomization tests of no effect in 4 comparisons. n = no restraint. ls = lap-shoulder belt.

	Restraint Use: (driver.passenger)			
	Same Use		Different Use	
Restraint Group	ls.ls	n.n	ls.n	n.ls
Number of Pairs	10996	3274	1412	1198
Mean Y_i	-0.059	0.061	-1.076	1.000
Standard error of mean	0.013	0.027	0.042	0.044
Standard deviation of Y_i	1.335	1.571	1.565	1.513
	Randomization tests			
	Huber Scores			
P-values	0.0000	0.0241	0.0000	0.0000
	Inner Trimmed Scores			
P-values	0.0000	0.0374	0.0000	0.0000

Sensitivity to nonrandomized treatment assignment

- Model says that, in the population prior to matching, treatment assignments are independent and two subjects with the same observed covariates may differ in their odds of treatment, $Z_{ij} = 1$, by at most a factor of Γ; then, the distribution of $\mathbf{Z}$ is returned to $\mathcal{Z}$ by conditioning on $\mathbf{Z} \in \mathcal{Z}$.

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- Equivalent to assuming that there is an unobserved covariate u_{ij} with $0 \leq u_{ij} \leq 1$ such that

$$\Pr\left(\mathbf{Z} = \mathbf{z} \mid \mathcal{F}, \mathcal{Z}\right) = \prod_{i \in \mathcal{I}} \frac{\exp\left(\gamma \sum_{j \in \mathcal{J}_i} z_{ij} \, u_{ij}\right)}{\sum_{j \in \mathcal{J}_i} \exp\left(\gamma \, u_{ij}\right)} = \frac{\exp\left(\gamma \mathbf{z}^T \mathbf{u}\right)}{\sum_{\mathbf{b} \in \mathcal{Z}} \exp\left(\gamma \mathbf{b}^T \mathbf{u}\right)},$$

for each $\mathbf{z} \in \mathcal{Z}$, where $\gamma = \log\left(\Gamma\right) \geq 0$; see Rosenbaum (2002, §4.2). For $\Gamma = 1$, $\gamma = \log\left(\Gamma\right) = 0$, this is the randomization distribution.

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- Distribution of $t\left(\mathbf{Z}, \mathbf{R}\right)$ under H_0 is unknown for $\Gamma > 1$ but the degree of departure from random assignment is controlled by the value of Γ.

Sensitivity analysis, continued

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- Similar for the lower bound, but with the two probabilities interchanged.
- Implementation for M-statistics in the `senm` and `senmCI` functions of the `R` package `sensitivitymult`.

Sensitivity analysis for Evan's comparison

Table: Upper bounds on P-values testing H_0.

	Same Use		Different Use	
Restraint Group	ls.ls	n.n	ls.n	n.ls
Γ	Huber Scores without Inner Trimming			
1	0.0000	0.0241	0.0000	0.0000
1.2	1.0000	1.0000	0.0000	0.0000
4			0.0000	0.0027
5			0.0211	0.4673
5.5			0.1808	1.0000
Γ	Inner Trimmed Scores			
1	0.0000	0.0374	0.0000	0.0000
1.2	1.0000	1.0000	0.0000	0.0000
5			0.0000	0.0125
6			0.0031	0.2219
6.5			0.0160	0.5058

A counterclaim

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- We will see that this counterclaim undermines itself.
- If this counterclaim were true, it would justify an analysis that is more insensitive to unmeasured bias than the analysis just performed.

A counterclaim analysis

- Suppose it were true that: *Seatbelts have no safety related effects, no effect on what happens during the accident. All we are seeing is a pattern produced by the type of person who wears safety belts.*

A counterclaim analysis

- Suppose it were true that: *Seatbelts have no safety related effects, no effect on what happens during the accident. All we are seeing is a pattern produced by the type of person who wears safety belts.*
- Were this true, it would justify an analysis confined to a segment of the data, not all of the pairs but just some of them.

A counterclaim analysis

- Suppose it were true that: *Seatbelts have no safety related effects, no effect on what happens during the accident. All we are seeing is a pattern produced by the type of person who wears safety belts.*
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- Will show the analysis, then explain why this analysis is licensed by the counterclaim.

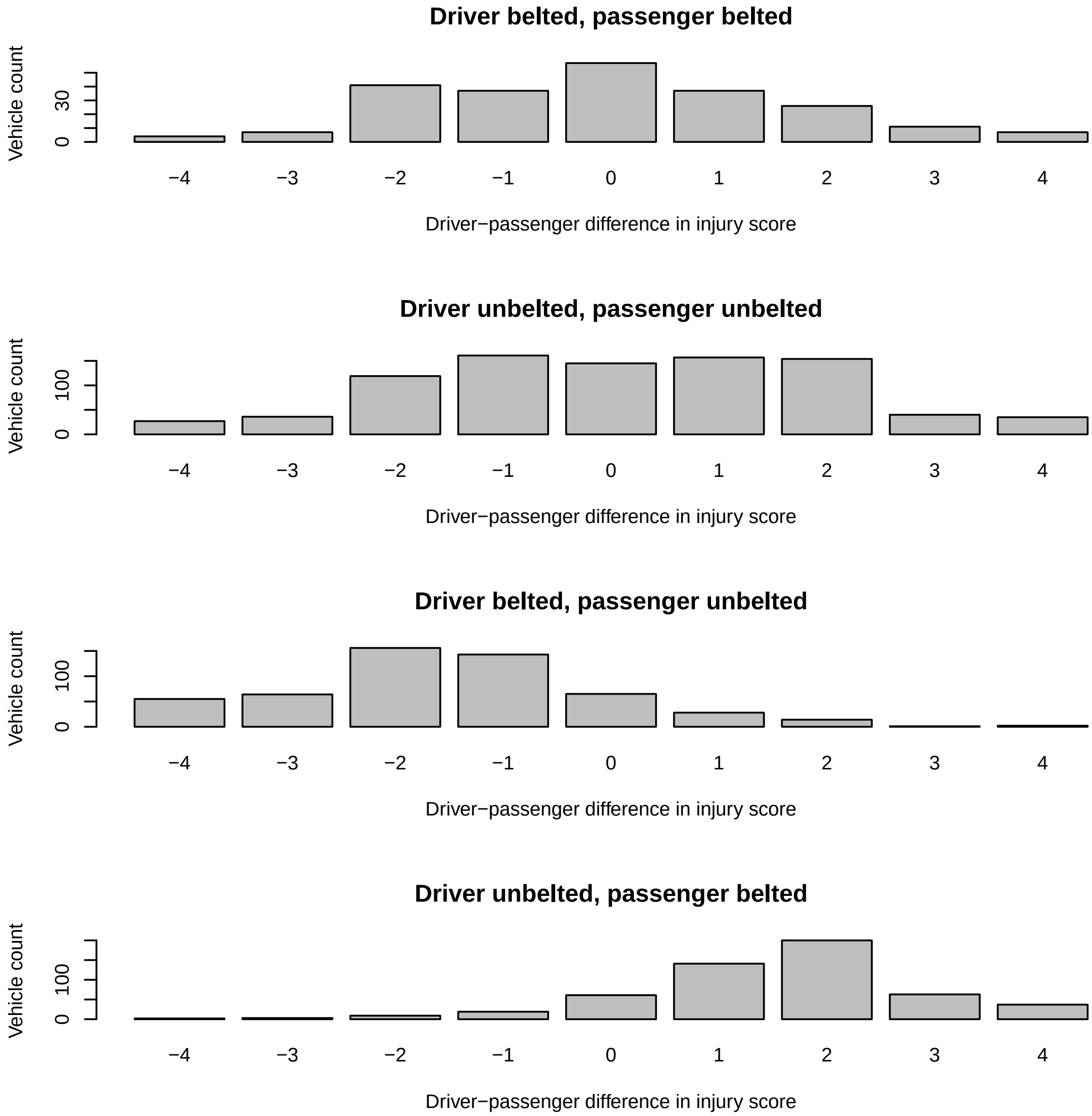

Figure 2: Pair differences in injury scores, driver-minus-passenger, for a driver and a passenger in the same car in FARS 2010-2011, by restraint use, when precisely one individual was ejected from the vehicle, either partially ejected or totally ejected. A positive difference indicates the driver suffered more severe injuries than the passenger.

Crashes with one ejection: Descriptive statistics

Table: Renalysis using only 2048 pairs in which exactly one person was ejected from the vehicle.

	Restraint Use: (driver.passenger)			
	Same Use		Different Use	
Restraint Group	ls.ls	n.n	ls.n	n.ls
Number of Pairs	222	782	522	522
Mean	-0.023	0.141	-1.540	1.584
Standard error	0.117	0.069	0.064	0.057
Standard deviation	1.748	1.938	1.455	1.291

Crashes with one ejection: Sensitivity analysis

Table: Values are upper bounds on P-values.

	Restraint Use: (driver.passenger)			
	Same Use		Different Use	
Restraint Group	ls.ls	n.n	ls.n	n.ls
Γ	Huber Scores without Inner Trimming			
1	0.7436	0.0428	0.0000	0.0000
1.2	1.0000	1.0000	0.0000	0.0000
9			0.0388	0.0009
11			0.2783	0.0149
Γ	Inner Trimmed Scores			
1	0.9002	0.0764	0.0000	0.0000
1.2	1.0000	0.8737	0.0000	0.0000
9			0.0047	0.0004
11			0.0322	0.0040

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- Example: if there are $n = 9$ subjects in matched triples, $\mathcal{J}_1 = \{1, 2, 3\}$, $\mathcal{J}_2 = \{1, 2, 3\}$, $\mathcal{J}_3 = \{1, 2, 3\}$, then one segment is $\mathcal{J}_1^{'} = \{2, 3\}$, $\mathcal{J}_2^{'} = \emptyset$, $\mathcal{J}_3^{'} = \{1, 2, 3\}$.

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- Let $\mathfrak{S}$ be the set whose 2^n elements are the 2^n possible segments.

Nondegenerate parts of a segment

- For a segment $\left\{\mathcal{J}_i', \, i \in \mathcal{I}\right\}$, write m_i for the random variable that counts the number of treated subjects in $\mathcal{J}_i'$, so $m_i = 0$ if $\mathcal{J}_i' = \emptyset$ and otherwise $m_i = \sum_{j \in \mathcal{J}_i'} Z_{ij}$, so $m_i = 0$ or $m_i = 1$. Write $\mathbf{m} = (m_1, \ldots, m_I)$.

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- The contribution from $\mathcal{J}_i'$ in segment $\left\{\mathcal{J}_i', \, i \in \mathcal{I}\right\}$ will be degenerate and uninteresting unless $m_i = 1 < \left|\mathcal{J}_i'\right|$, that is, unless $\mathcal{J}_i'$ contains the treated subject and at least one control from matched set $\mathcal{J}_i$.

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- For matched pairs, $|\mathcal{J}_i| = J_i = 2$ for all i, nondegenerate part of a segment is a subset of the matched pairs.
- For matched sets with $|\mathcal{J}_i| = J_i > 2$, a segment $\left\{\mathcal{J}_i',\, i \in \mathcal{I}\right\}$ may have nondegenerate parts $\mathcal{J}_i'$ with $m_i = 1 < \left|\mathcal{J}_i'\right| < |\mathcal{J}_i|$ containing the treated subject from $\mathcal{J}_i$ and some but not all of the controls from $\mathcal{J}_i$.

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- As before, conditioning on the event $\mathbf{Z}' \in \mathcal{Z}_{\mathbf{m}}'$ is abbreviated as conditioning on $\mathcal{Z}_{\mathbf{m}}'$, and generally the conditioning will be on $(\mathcal{Z}, \mathcal{Z}_{\mathbf{m}}', \mathbf{m})$ jointly.

Using a matrix of data to determine a segment

- There is a $n \times M$ matrix $\mathbf{W}$ describing with row $\mathbf{w}_{ij}$ describing subject ij. Write $\mathcal{W}$ for the set of possible values for $\mathbf{W}$.

Definition

The phrase "$\mathbf{W}$ determines the segment" means that there is a known function $\mathcal{S}(\mathbf{W})$ that receives $\mathbf{W}$ and returns a segment from $\mathfrak{S}$, that is, $\mathcal{S} : \mathcal{W} \to \mathfrak{S}$.

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- Unless $\mathbf{W}$ includes $\mathbf{Z}$, a segment determined by $\mathbf{W}$ cannot make use of the identity of the treated subject.

A basic question about analysis of a segment

- When can we select a segment $\left\{\mathcal{J}_i', \, i \in \mathcal{I}\right\}$ using $\mathbf{W}$, yet appropriately analyze this segment as if were an unselected data set?

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- **Corollary**: If the sensitivity model governs treatment assignment, if a segment $\mathcal{S}(\mathbf{S}) = \left\{ \mathcal{J}_i^{'},\ i \in \mathcal{I} \right\}$ is determined by the observed value of the supplementary responses $\mathbf{S}$, and if the supplementary responses are unaffected by the treatment, $\mathbf{s}_{Tij} = \mathbf{s}_{Cij}$ for all ij, then the distribution of treatment assignments in the segment is given by (1).

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- By the corollary, this licenses an analysis focused on the segment of crashes with one ejection.
- Expressed informally, the counterclaim said the unbelted individual was injured because he was frail, but switching treatment assignments (i.e., belting him) would have changed the identity of the belted subject but would have changed no safety outcomes.

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- Depending upon the context, this concession acknowledging that the treatment does cause an effect on $(\mathbf{s}_{Tij}, \mathbf{s}_{Cij})$ while denying an effect on (r_{Tij}, r_{Cij}) may be a large concession.

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- Another supplementary outcome is of direction of initial impact.
- Will look at crashes in which there was one ejection and the initial impact was not from the side. (That is, the initial impact was front or rear or unknown.)
- Might be the case that an important source of variation in injury is whether you are seated on the side of the initial impact.

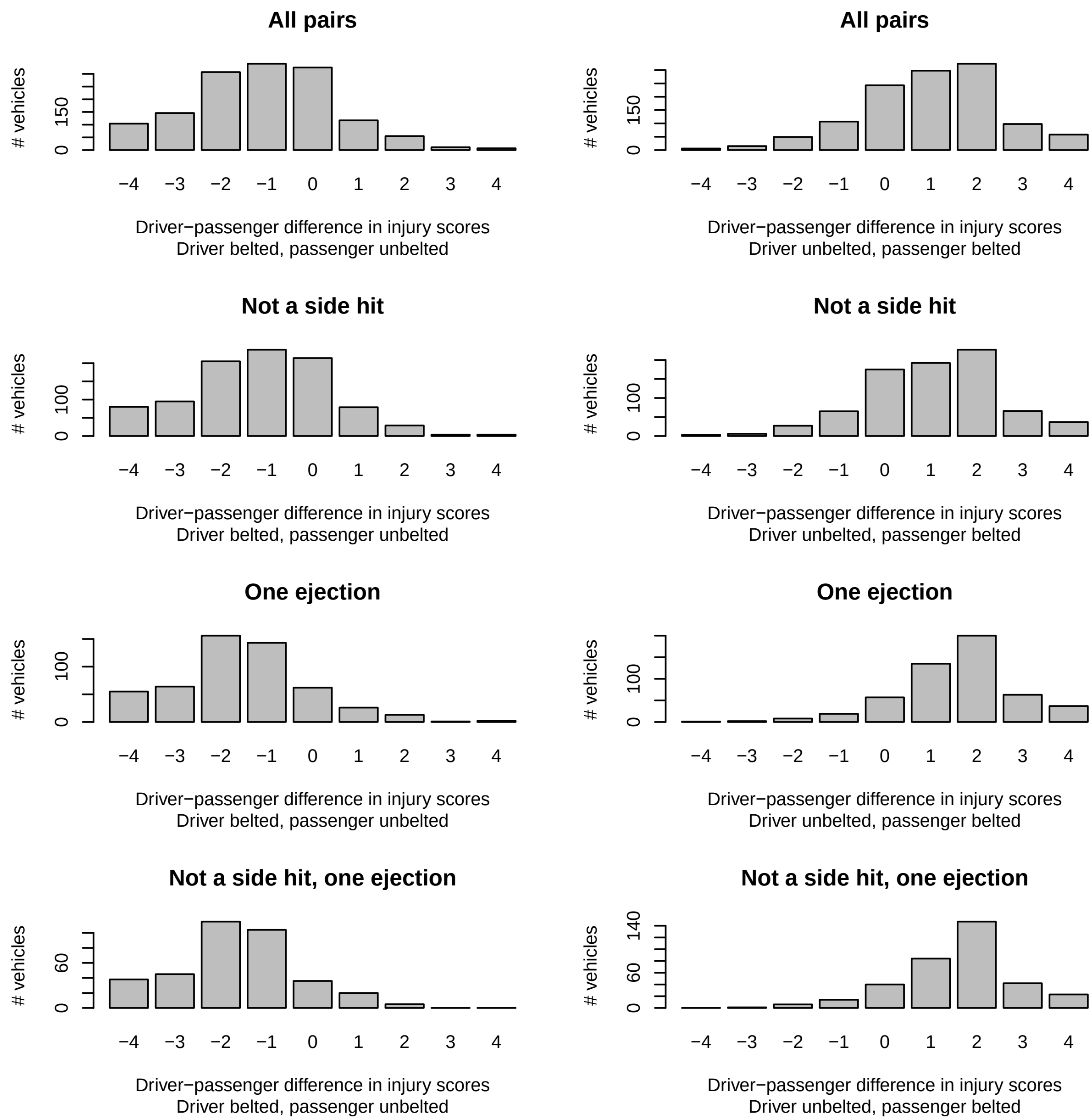

Figure 3: Pair differences in injury scores, driver-minus-passenger, for a driver and a passenger in the same car in FARS 2010-2011, by restraint use, for all vehicle pairs, for vehicles not known to have an initial collision from the side, for vehicles with exactly one ejection, and for vehicles not know to have an initial collision from the side with exactly one ejection. A positive difference indicates the driver suffered more severe injuries than the passenger.

One ejection, not a side hit: Descriptive statistics

Table: Renalysis of differences in injury scores using only 1383 pairs in which exactly one person was ejected from a vehicle whose initial impact was not from the side. n = no restraint. ls = lap-shoulder belt.

	Restraint Use: (driver.passenger)			
	Same Use		Different Use	
Restraint Group	ls.ls	n.n	ls.n	n.ls
Number of Pairs	153	510	363	357
Mean	-0.072	0.133	-1.628	1.588
Standard error	0.145	0.087	0.071	0.067
Standard deviation	1.789	1.961	1.345	1.259

One ejection, not a side hit: Sensitivity analysis

Table: Upper bounds on P-values.

Restraint Group	ls.ls	n.n	ls.n	n.ls
Number of Pairs	153	510	363	357
Γ	Huber Scores without Inner Trimming			
1	0.6182	0.1251	0.0000	0.0000
1.2	1.0000	1.0000	0.0000	0.0000
11			0.0291	0.0291
12			0.0610	0.0614
15			0.2722	0.2774
Γ	Inner Trimmed Scores			
1	0.8788	0.1729	0.0000	0.0000
1.2	1.0000	0.9732	0.0000	0.0000
15			0.0129	0.0439

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- Design sensitivity $\widetilde{\Gamma}$ depends on the process that generated the data (sampling model) and on the methods of analysis.
- Design sensitivity $\widetilde{\Gamma}$ is computed under a simple model with a treatment effect and no unmeasured bias.
- Design sensitivity $\widetilde{\Gamma}$ is a measure of our ability to distinguish two sharply distinct situations: (i) biased treatment assignment with no treatment effect, H_0, and (ii) a genuine treatment effect (H_0 is false) and no unmeasured bias (random assignment of treatments).

Simple model for injury and ejection, part 1

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- Injury model

$$r_{Tij} = r_{Cij} + \tau + \beta\,(s_{Tij} - s_{Cij})$$

so $r_{Tij} - r_{Cij} = \tau$ if the treatment does not affect whether you are ejected, or $r_{Tij} - r_{Cij} = \tau + \beta$ if the treatment (e.g., being unbelted) causes you to be ejected, $(s_{Tij}, s_{Cij}) = (1,0)$.

Simple model for injury and ejection, part 2

- Injury model

$$r_{Tij} = r_{Cij} + \tau + \beta \left(s_{Tij} - s_{Cij} \right)$$

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- Will look at this for $\varepsilon_i \sim N(0, 1)$, and randomized treatment assignment, $\Pr\left(\mathbf{Z} = \mathbf{z} \mid \mathcal{F}, \mathcal{Z} \right) = 2^{-I}$ for each $\mathbf{z} \in \mathcal{Z}$. Results are similar with logistic errors.

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- Will set $\beta = \left(\frac{1}{2} - \tau\right)/\pi_{10}$ so that $\mathrm{E}\left(Y_i\right) = \frac{1}{2}$ in all cases.

Design sensitivities under the simple Normal model

Table: Design sensitivities using all pairs (All), the segment (Seg), and its complement (Comp), without or with inner trimming. The largest design sensitivities in each row are in **bold**.

	No inner trim, ψ_{hu}			With inner trim, ψ_{in}		
	$(\pi_{11}, \pi_{10}, \pi_{00}) = (1/3, 1/3, 1/3)$					
τ	All	Seg	Comp	All	Seg	Comp
0	2.7	3.3	2.2	3.8	**4.9**	2.8
1/4	3.2	3.6	2.8	4.4	**5.1**	3.7
1/2	3.4	3.4	3.4	**4.7**	**4.7**	**4.7**
	$(\pi_{11}, \pi_{10}, \pi_{00}) = (1/4, 1/2, 1/4)$					
τ	All	Seg	Comp	All	Seg	Comp
0	3.0	3.8	2.1	4.0	**5.3**	2.5
1/4	3.3	3.8	2.7	4.5	**5.3**	3.5
1/2	3.5	3.5	3.5	**4.8**	**4.8**	**4.8**

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- Becomes Fisher's method for combining P-values when $\kappa = 1$.
- Hsu et al. (2013) evaluate the truncated product in sensitivity analyses, finding $\kappa = 0.2$ is better than $\kappa = 1$.

Simulated power of a 0.05-level sensitivity analysis

Table: Power of a 0.05-level sensitivity analysis at $\Gamma = 4$, using all $I = 2000$ pairs (All), the segment (Seg), its complement (Comp), and the truncated product (Tprod), $\kappa = 0.2$, based on both the segment and its complement, using inner trimming. I_{Seg} is the expected number of pairs in the segment.

Distribution	τ	I_{Seg}	All	Seg	Comp	Tprod
$(\pi_{11}, \pi_{10}, \pi_{00}) = (1/3, 1/3, 1/3)$						
Normal	0	1111	0.01	0.48	0.00	0.22
Normal	1/4	1111	0.24	0.62	0.01	0.38
Normal	1/2	1111	0.61	0.40	0.33	0.55
$(\pi_{11}, \pi_{10}, \pi_{00}) = (1/4, 1/2, 1/4)$						
Normal	0	1250	0.04	0.82	0.00	0.60
Normal	1/4	1250	0.39	0.80	0.00	0.60
Normal	1/2	1250	0.60	0.43	0.29	0.54

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- Such a counterclaim fails in its role as a counterclaim. Supposing it to be true would only strengthen the evidence in support of the original claim.
- An investigator may examine potential counterclaims before they are raised by critics.
- Design sensitivities and simulated powers of sensitivity analyses suggest that what occurred in the example is expected under certain simple models for an effect without bias.

Proof of the proposition

The segment $\left\{\mathcal{J}_i^{'},\ i \in \mathcal{I}\right\}$ is fixed by conditioning on $\mathcal{F}$; moreover, the set $\mathcal{Z}'_{\mathbf{m}}$ is a fixed set as a consequence of conditioning on $\mathcal{Z}$ and $\mathbf{m}$. It suffices to consider a single set i. If $\mathcal{J}_i^{'}$ is degenerate, then it contributes a 1 factor to distribution in the segment. Otherwise, for $\left|\mathcal{J}_i^{'}\right| \geq 2$ and $m_i = 1$, the conditional probability that $Z_{ij} = z'_{ij}$ for $j \in \mathcal{J}_i^{'}$ given $\mathcal{F}$, $\mathcal{Z}$, $\mathcal{Z}'_{\mathbf{m}}$, $\mathbf{m}$ is the ratio of $\exp\left(\gamma \sum_{j \in \mathcal{J}_i} z_{ij}^{'} u_{ij}\right) / \sum_{j \in \mathcal{J}_i} \exp\left(\gamma\, u_{ij}\right)$ to the sum of similar terms over $j \in \mathcal{J}_i^{'}$, namely

$$\frac{\exp\left(\gamma \sum_{j \in \mathcal{J}_i} z_{ij}^{'} u_{ij}\right) / \sum_{j \in \mathcal{J}_i} \exp\left(\gamma\, u_{ij}\right)}{\sum_{j \in \mathcal{J}_i^{'}} \left\{\exp\left(\gamma\, u_{ij}\right) / \sum_{j \in \mathcal{J}_i} \exp\left(\gamma\, u_{ij}\right)\right\}} = \frac{\exp\left(\gamma \sum_{j \in \mathcal{J}_i^{'}} z_{ij}^{'} u_{ij}\right)}{\sum_{j \in \mathcal{J}_i^{'}} \exp\left(\gamma\, u_{ij}\right)}$$

as in the statement of the proposition.