Component-Level Redundancy is Better Than System-Level Redundancy for Channel Graphs

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A well known engineering principle in introducing redundancy to a system is that component-level redundancy is better than system-level redundancy. However, the mathematical demonstration of this principle is usually for coherent systems constructed by one sequential synthesis and one parallel synthesis (the proof consists of showing that the parallel synthesis should be at the component level). In this article we demonstrate the same principle for systems constructed by parallel-sequential-parallel synthesis, one parallel synthesis at the component level and one at the system level. We show that the parallel synthesis at the component level should have more redundancy. The particular vehicle we use for this demonstration is the comparison of blocking probabilities for a 2-port network known as channel graphs. The crux of the mathematical demonstration is the lemma that \( [1 - \prod_{i=1}^{n} (1 - p_i)]^{1/n} \) decreases in \( a, 0 < p_i < 1, i = 1, \ldots, n \).

1. INTRODUCTION

An \( s \)-stage channel graph is a graph whose vertices can be partitioned into \( s \) subsets (stages) \( V_1, V_2, \ldots, V_s \), with \( V_1 \) and \( V_s \) each containing a single vertex (called the source and the sink, respectively), and whose edges can be partitioned into \( s - 1 \) subsets \( E_1, E_2, \ldots, E_{s-1} \) such that edges in \( E_i \) connect vertices in \( V_i \) to vertices in \( V_{i+1} \).

Channel graphs have been extensively used in modeling and analyzing blocking probabilities of communication networks. A very popular model (due to its simplicity) is to assume

(i) Each edge can be in one of two states, occupied or idle, and the states of the edges are independent.

(ii) Each edge in \( E_i \) has probability \( p_i \), called the occupancy for \( E_i \), of being occupied.

The blocking probability of a channel graph is the probability that every channel—by which we mean a path from source to sink consisting of one edge from each \( E_i \)—
contains at least one occupied edge. An s-stage channel graph is said to be superior to another s-stage channel graph if the blocking probability of the former never exceeds that of the latter, independent of the occupancies for the $E_i$ (common to both graphs).

In this paper we are concerned with the case that each $E_i$ consists of $n$ edges. For $m$ dividing $n$ let $G_{i,s}(m)$ denote the channel graph which can be decomposed into $n/m$ isomorphic $s$-stage channel graphs such that each such graph has one vertex in each stage and $m$ edges connecting any two vertices in adjacent stages. We show $G_{3,8}(6)$, $G_{3,8}(3)$, $G_{3,8}(2)$, and $G_{3,8}(1)$ in Figure 1.

Chung and Hwang [2] proved that $G_{s,n}(m)$ is superior to $G_{s,n}(m')$ whenever $m'$ divides $m$. In particular, this implies that $G_{s,n}(n)$ is superior to any $G_{s,n}(m)$ which, in turn, is superior to $G_{s,n}(1)$. In this paper we prove that $G_{s,n}(m)$ is superior to $G_{s,n}(m')$ whenever $m > m'$. A verbal interpretation of this result is that component-level redundancy is better than system-level redundancy.

2. THE MAIN RESULT

We first prove a probability inequality which is of interest itself and is also crucial in proving our main result.

Lemma. $[1 - \prod_{i=1}^{k} (1 - p_i^a)]^{1/a}$ is monotone decreasing in $a$.

Proof. Define

$$f(a) = \frac{1}{a} \ln \left[ 1 - \prod_{i=1}^{k} (1 - p_i^a) \right]$$

Then it suffices to prove that $f(a)$ is monotone decreasing in $a$.

$$f'(a) = -\frac{1}{a^2} \ln \left[ 1 - \prod_{i=1}^{k} (1 - p_i^a) \right] + \prod_{i=1}^{k} (1 - p_i^a) \sum_{j=1}^{k} \frac{p_i^a \ln p_i}{1 - p_i^a}$$

$$= \frac{1}{a^2} \left\{ -\ln \left[ 1 - \prod_{i=1}^{k} (1 - p_i^a) \right] + \prod_{i=1}^{k} (1 - p_i^a) \sum_{j=1}^{k} \frac{p_i^a \ln p_i}{1 - p_i^a} \right\}$$

$$= \frac{1}{a^2} \left\{ -\ln (1 - e^{-\sum_{i=1}^{k} \theta_i}) + \sum_{i=1}^{k} \frac{(1 - e^{-\theta_i}) \ln (1 - e^{-\theta_i})}{1 - e^{\sum_{i=1}^{k} \theta_i}} \right\}$$

$$= \frac{e^{-\sum_{i=1}^{k} \theta_i}}{a^2 (1 - e^{-\sum_{i=1}^{k} \theta_i})} \left\{ -\frac{(1 - e^{-\sum_{i=1}^{k} \theta_i}) \ln (1 - e^{-\sum_{i=1}^{k} \theta_i})}{e^{-\sum_{i=1}^{k} \theta_i}} \right\}$$

$$+ \sum_{j=1}^{k} \frac{(1 - e^{-\theta_j}) \ln (1 - e^{-\theta_j})}{e^{-\theta_j}}$$.
where $e^{-\theta} = 1 - p_i$ by definition (this implies $\theta_i \geq 0$). Define

$$g(\theta_1, \ldots, \theta_k) = \sum_{j=1}^{k} \frac{(1 - e^{-\theta_j}) \ln (1 - e^{-\theta_j})}{e^{-\theta_j}} = \sum_{j=1}^{k} (e^{-\theta_j} - 1) \ln (1 - e^{-\theta_j}).$$

Clearly, $g$ is symmetric with respect to $\theta_j$. We will also prove that

$$(\theta_i - \theta_j) \left[ \frac{\partial g}{\partial \theta_i} - \frac{\partial g}{\partial \theta_j} \right] \geq 0 \quad \text{for any } i, j$$

Therefore $g$ is a Schur function [6] and

$$g(\theta_1, \ldots, \theta_k) \geq g(\theta'_1, \ldots, \theta'_k)$$

if $(\theta_1, \ldots, \theta_k)$ majorizes $(\theta'_1, \ldots, \theta'_k)$, i.e.,

$$\sum_{i=1}^{j} \theta_i \geq \sum_{i=1}^{j} \theta'_i \quad \text{for every } j = 1, \ldots, k - 1$$

$$\sum_{i=1}^{k} \theta_i = \sum_{i=1}^{k} \theta'_i$$

Consequently, given $\sum_{i=1}^{k} \theta_i = \theta$, $g(\theta_1, \ldots, \theta_k)$ attains its maximum at $(\theta, 0, \ldots, 0)$. It follows that

$$f'(a) = \frac{e^{-\theta}}{a^{\gamma}(1 - e^{-\theta})} \left\{ -g(\theta, 0, \ldots, 0) + g(\theta_1, \ldots, \theta_k) \right\} \leq 0$$

Next we prove that

$$(\theta_i - \theta_j) \left[ \frac{\partial g}{\partial \theta_i} - \frac{\partial g}{\partial \theta_j} \right] \geq 0$$

to complete the proof.

$$\frac{\partial g}{\partial \theta_j} = e^{\theta} \ln (1 - e^{-\theta}) + \frac{(e^{\theta} - 1)e^{-\theta}}{1 - e^{-\theta}}$$

$$= e^{\theta} \ln (1 - e^{-\theta}) + 1$$

Define

$$h(\theta) = e^{\theta} \ln (1 - e^{-\theta})$$
Then
\[ h'(\theta) = e^\theta \ln(1 - e^{-\theta}) + \frac{1}{1 - e^{-\theta}} \]
\[ = e^\theta \left[ \ln(1 - e^{-\theta}) + \frac{1 - (1 - e^{-\theta})}{1 - e^{-\theta}} \right] \]
\[ = e^\theta \left[ -\ln(1 - e^{-\theta}) + \frac{1}{1 - e^{-\theta}} - 1 \right] \geq 0 \]
since \( x - 1 \geq \ln x \). Therefore
\[ (\theta_i - \theta_j) \left[ \frac{\partial g}{\partial \theta_i} - \frac{\partial g}{\partial \theta_j} \right] \geq 0. \]

**Corollary.** \( [1 - \Pi_{i=1}^s (1 - p_i^{m})]^m \) is monotone decreasing in \( m \).

It is easily verified that the blocking probability of \( G_{s,n}(m) \) is exactly the term in the Corollary with \( k = s - 1 \). Therefore we have

**Theorem.** \( G_{s,n}(m) \) is superior to \( G_{s,n}(m') \) for \( m > m' \).

### 3. SOME CONCLUDING REMARKS

We mention here some results in the literature which are relevant to our result.

Hwang and Odlyzko [5] proved that \( G_{s,n}(1) \) is superior to \( G_{s,n}(n) \), an interesting contrast to our result that \( G_{s,n}(n) \) is superior to \( G_{s,n}(m) \) for any \( m < n \). Several other papers [1–4] have given sufficient conditions for comparing the superiority of two channel graphs constructed by series-parallel syntheses. Though the channel graphs discussed in this paper are also constructed by series-parallel syntheses, our result cannot be deduced from any of the known conditions.

The restriction that the parameter \( m \), which represents the degree of component-level parallelism, is constant throughout the stages is not as restrictive as it sounds since an edge in a given stage \( E_i \) can itself be the condensation of a channel graph as long as each edge in \( E_i \) is such a condensation and \( p_i \) is the correct blocking probability.

**References**


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