Statistics 430
HW #1 Solutions

Thanks to Emil Pitkin

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1 Ch 1 Problem 10

a

If there are no restrictions on the seating arrangement, we permute all eight positions: $\boxed{8! = 40{,}320}$

b

Because they can't be separated, consider AB to be one person.[1] There are now 7 "people" to arrange, and they can be permuted in 7! ways. Since A and B could be arranged as AB or BA, we multiply the preceding by 2! : $\boxed{7!2! = 10,080}$.

c

The men and women must alternate seats. Under one arrangement, seats 1,3,5,7 are reserved for the men, and 2,4,6,8 for the women. You can permute the males among the male seats and the women among the female seats in 4!4! ways. Because the men could have occupied the even seats, and the women the odd seats, multiply by 2: $\boxed{2 \times 4!4! = 1152}$.

d

The 5 men are in a chain gang and can have no women between them, though their order in the gang is not fixed. You can permute the 5 men, as well as the 3 women in the remaining seats in 5!3! ways. The left-most of the 5 men can occupy seat 1, 2, 3, or 4, so multiply the preceding by 4: $\boxed{4 \times 5!3! = 2{,}880}$.

e

Again, consider each married couple as a super glued group. The 4 groups can be permuted in 4! ways. In each group, the husband can sit in one of two ways – either to the right of his wife or to the left. The answer is therefore $\boxed{2^4 \times 4! = 384}$.

[1] A la American Pie, super glue A and B together

2 Ch 1 Problem 16

a

The student can either sell a pair of math books ($\binom{6}{2} = 15$ possible pairs), a pair of science books ($\binom{7}{2} = 21$ possible pairs), or a pair of econ books ($\binom{4}{2} = 6$ possible pairs). Summing, we see the answer is $\boxed{42}$.

b

The student must choose either one math and one science textbook ($\binom{6}{1} \times \binom{7}{1} = 42$), one science and one econ ($\binom{7}{1} \times \binom{4}{1} = 28$) , or one math and one econ ($\binom{6}{1} \times \binom{4}{1} = 24$). After summing, the answer is $\boxed{94}$.

3 Ch 1 Problem 19

a

Were the men on good terms, any of the $\binom{6}{3}$ groups of 3 could be chosen to serve on the committee. But 2 are bickering, disqualifying any group where the two of them are chosen to serve together. How many such groups are there? $\binom{2}{2} \times \binom{4}{1}$, since in the disallowed groups both of the bickering men must be chosen, as well as one of the remaining 4 friendly men. Because the women are on good terms with one another, any 3 of them (out of 8) can comprise the female group on the committee. The answer is therefore $\boxed{(\binom{6}{3} - \binom{2}{2} \times \binom{4}{1}) \times \binom{8}{3} = 896}$.

b

Argue analogously to (a). $\boxed{(\binom{8}{3} - \binom{2}{2} \times \binom{6}{1}) \times \binom{6}{3} = 1000}$.

c

A similar flavor to a and b. If the world could just get along, $\binom{8}{3} \times \binom{6}{3}$ committees could be formed. But one man and one woman cannot get along, disqualifying all possible committees on which they are called to serve. On such a committee, that man, that woman, 2 friendly men ($\binom{5}{2}$ choices) and 2 friendly women ($\binom{6}{2}$ choices) serve. The answer is therefore $\boxed{\binom{8}{3} \times \binom{6}{3} - \binom{7}{2} \times \binom{5}{2} = 910}$.

4 Ch 1 Problem 23

The 3 sets of twins can be assigned to the 3 rooms in 3! ways. Within each of the 3 rooms, a pair of twins can lie on the 2 beds in 2! different ways. The answer is therefore $\boxed{3! \times (2!)^3 = 48}$

5 Ch 1 Problem 31

a

The question can be reformulated. How many nonnegative solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 8$? x_i corresponds to the number of blackboards in the ith school, with the total number summing to 8. We know from lecture that the answer is $\boxed{\binom{8+4-1}{4-1}}$. The expression equals $\boxed{165}$.

b

Again, reformulate. How many positive solutions are there to the equation $x_1+x_2+x_3+x_4 = 8$? x_i corresponds to the number of blackboards in the ith school, with the total number summing to 8. We know from lecture that the answer is $\boxed{\binom{8-1}{4-1}}$. The expression equals $\boxed{35}$.

6 Ch 1 Theoretical Problem 4

First imagine that the r black and n-r white balls are distinguishable, i.e., it makes sense to identify "black ball number 3." There would then be $n!$ arrangements. But the r black balls are actually unnumbered, so all $r!$ permutations of the black balls are indistinguishable; likewise for the $(n-r)!$ permutations of the white balls. Therefore divide $n!$ by $r!(n-r)!$. Indeed, $\frac{n!}{r!(n-r)!} = \binom{n}{r}$.

7 Ch 1 Theoretical Problem 8

Follow the hint. Out of n men and m women, $\binom{n+m}{r}$ groups of size r can be chosen. The group can be composed of no men and r women ($\binom{n}{0}\binom{m}{r}$ such groups), of 1 man and $r-1$ women ($\binom{n}{1}\binom{m}{r-1}$ such groups), and generally, of i men and $r-i$ women ($\binom{n}{i}\binom{m}{r-i}$ such groups). Adding all such possible groups yields the desired expression, written compactly as $\sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i}$.

8 Ch 2 Problem 12

a

This is a classic inclusion-exclusion problem. Denoting French by F, etc., we are given that P(F)=.26, P(S)=.28, P(G)=.16, P(S∩F)=.12, P(S∩G)=.04, P(F∩G)=.06, P(S∩F∩G)=.02. The event "not in any class" is the complement of the event S∪F∪G, i.e, "in at least one class," whose probability is P(S∪F∪G) = P(S)+P(F)+P(G)-(P(S∩F)+ P(S∩G) + P(F∩G)) + P(S∩F∩G) = .5. So the probability of not taking any class is 1-.5=$\boxed{.5}$

b

As an illustrative example, we calculate the number of students taking only Spanish. We count everybody who is taking Spanish (28), subtract the number of those taking Spanish

AND French (-12), subtract those taking Spanish AND German (-4), and add back those taking all three languages (2), since they were subtracted twice (as part of the Spanish and French, and Spanish and German groups). Arguing similarly for the other languages, we see the number of students taking one language only is 28-(12+4)+2 + 26-(12+6)+2 + 16-(4+6)+2=32. The corresponding probability, since there are 100 students to choose from, is $\boxed{.32}$.

c

The probability that at least 1 of the 2 students is taking a language class is 1 minus the probability that neither of the 2 students is taking a language class. In part (a) we found the number of students not taking any of the classes to be 50. So the probability that 2 randomly selected students, out of 100, belong to this group of 50 slackers, is $\frac{\binom{50}{2}}{\binom{100}{2}} = \frac{49}{98}$. 1 minus this quantity equals $149/198 = .7525$.

9 Ch 2 Problem 23

There are 36 possible outcomes when two dice are rolled. Count on your fingers, for each possible roll of a die, the number of ways a second die could exceed it. Answer = $\frac{5+4+3+2+1+}{36} = \boxed{\frac{5}{12} = .417}$.

10 Ch 2 Problem 25

Follow the hint, and let E_n denote the event "the dice roll sequence ends because a 5 has been rolled on the nth roll"; implicit is the understanding that no 5 or 7 has been rolled in the first n-1 rolls. What two conditions need to be met in order for E_n to occur?

1. In each of the first n-1 (independent) rolls, no 5 or 7 appears. On any given roll, the probability of rolling 5 OR 7 is $\frac{4}{36} + \frac{6}{36} = \frac{10}{36}$, so the probability of rolling neither 5 nor 7 on some roll is $1 - \frac{10}{36} = \frac{26}{36}$. The probability of not rolling a 5 or 7 for n-1 consecutive rolls is therefore $\left(\frac{26}{36}\right)^{n-1}$
2. A 5 is rolled on the nth roll. This happens with probability $\frac{4}{36}$.

We combine (1) and (2) to find that the probability of E_n is $\frac{4}{36} \times (\frac{26}{36})^{n-1}$. Our ultimate goal is to find $P(\bigcup_{n=1}^{\infty} E_n)$, the probability that the sequence of rolls is terminated with a 5, eventually. Notice that the E_n are disjoint, since the sequence can only be terminated at one particular roll. By the third axiom of probability, then, $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$.

Putting our thoughts and observations together:

$$\begin{aligned}
\mathrm{P}\left(\bigcup_{n=1}^{\infty} E_n\right) &= \sum_{n=1}^{\infty} P(E_n) \\
&= \frac{4}{36}\sum_{n=1}^{\infty}\left(\frac{26}{36}\right)^{n-1} \\
\text{now reindex:} \quad &= \frac{4}{36}\sum_{k=0}^{\infty}\left(\frac{26}{36}\right)^{k} \\
\text{remember the properties of geometric series:} \quad &= \frac{4}{36} \times \frac{1}{1-\frac{26}{36}} \\
&= \boxed{\tfrac{4}{10} = .4}
\end{aligned}$$

11 Ch 2 Problem 35

a

$$\boxed{\frac{\binom{12}{3}\binom{16}{2}\binom{18}{2}}{\binom{46}{7}} = .075}$$

b

Calculate 1 minus the probability of either 1 or 2 withdrawn red balls:

$$\boxed{1\text{-}\left(\frac{\binom{12}{0}\binom{34}{7} + \binom{12}{1}\binom{34}{6}}{\binom{46}{7}}\right) = 0.598}$$

c

$$\boxed{\frac{\binom{12}{7} + \binom{16}{7} + \binom{18}{7}}{\binom{46}{7}} = .00082}$$

d

If you want exactly 3 red balls, then you have to eliminate those scenarios when 3 blue balls (and 1 green ball) are drawn as well.

$$\boxed{\frac{\binom{12}{3}\left(\binom{34}{4} - \binom{16}{3}\binom{18}{1}\right) + \binom{16}{3}\left(\binom{30}{4} - \binom{12}{3}\binom{18}{1}\right)}{\binom{46}{7}} = .3945}$$

12 Ch 2 Problem 49

There are 6 men. In order for both groups to have the same number of men, each of the groups must be composed of 3 men (and, therefore, of 3 women as well). Now imagine seating 12 people side by side, where each person is an anonymous M or W. Also imagine that after seating we erect a partition between the 6 on the left end and the 6 on the right. Let's direct our attention to the gentlemen. There are $\binom{12}{6}$ ways of seating the 6 M in the

12 seats. There are $\binom{6}{3}$ of seating 3 M to the left of the partition, and $\binom{6}{3}$ ways of seating 3 M to the right of the partition. There is just 1 way for the 6 W to take the remaining seats. The answer is therefore $\frac{\binom{6}{3}^2}{\binom{12}{6}} = .4329$.

13 Ch 2 Theoretical Problem 4

As we often do, we prove that $A = B$ by showing that $A \subset B$ and that $B \subset A$.

i

Suppose x is an outcome of $(\bigcup_{i=1}^{\infty} E_i) \cap F$. Then x is contained in at least some E_i, as well as in F. So x is contained in $E_i \cap F$, and therefore in $\bigcup_{i=1}^{\infty}(E_i \cap F)$. Hence $\boxed{(\bigcup_{i=1}^{\infty} E_i) \cap F \subset \bigcup_{i=1}^{\infty}(E_i \cap F)}$. Proceeding in the other direction, suppose x is an outcome of $\bigcup_{i=1}^{\infty}(E_i \cap F)$. Then x is contained in $E_i \cap F$ for some i, therefore both in E_i for that i, and in F. Since it is contained in E_i it is contained in $(\bigcup_{i=1}^{\infty} E_i)$. Since it is contained in F as well, x lies in $(\bigcup_{i=1}^{\infty} E_i) \cap F$. We have thus also shown that $\boxed{\bigcup_{i=1}^{\infty}(E_i \cap F) \subset (\bigcup_{i=1}^{\infty} E_i) \cap F}$, so we are done.

ii

Suppose x is an outcome of $(\bigcap_{i=1}^{\infty} E_i) \cup F$. Then x is contained in all of the E_i, or in F, or in all of the above – which is to say that for each i, x is either in E_i, or F, or both, i.e., x lies in $\bigcap_{i=1}^{\infty}(E_i \cup F)$. We have shown that $\boxed{(\bigcap_{i=1}^{\infty} E_i) \cup F \subset \bigcap_{i=1}^{\infty}(E_i \cup F)}$. Going in the other direction, assume x is an outcome of $\bigcap_{i=1}^{\infty}(E_i \cup F)$. Then for each i, x is a member of E_i, or F, or both – which is equivalent to x's being a member of all the E_i, or F, or both, i.e., lying in $\bigcap_{i=1}^{\infty}(E_i \cup F)$. Indeed, $\boxed{(\bigcap_{i=1}^{\infty}(E_i \cup F \subset \bigcap_{i=1}^{\infty} E_i) \cup F)}$.

14 Ch 2 Theoretical Problem 11

We show the general case, from which the particular example follows directly.

$$\begin{aligned} 1 &\geq P(E \cup F) \\ 1 &\geq P(E) + P(F) - P(E \cap F) \\ P(E \cap F) &\geq P(E) + P(F) - 1 \end{aligned}$$

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15 Ch 2 Theoretical Problem 16

The base case is trivial.

Induction hypothesis: assume that for $1 \leq k \leq n-1$:

$$P(E_1 \cap E_2 \cap \ldots \cap E_k) \geq P(E_1) + \ldots + P(E_k) - (k-1)$$

Then

$$
\begin{aligned}
1 &\geq P(E_1 \cap E_2 \cap \ldots \cap E_{n-1} \cup E_k) \\
1 &\geq P(E_1 \cap E_2 \cap \ldots \cap E_{n-1}) + P(E_k) - P(E_1 \cap E_2 \cap \ldots \cap E_{n-1} \cap E_k) \\
P(E_1 \cap E_2 \cap \ldots \cap E_{n-1} \cap E_n) &\geq P(E_1 \cap E_2 \cap \ldots \cap E_{n-1}) + P(E_n) - 1 \\
P(E_1 \cap E_2 \cap \ldots \cap E_n) &\geq P(E_1) + \ldots + P(E_{n-1}) - ((n-1)-1) + P(E_n) - 1 \\
P(E_1 \cap E_2 \cap \ldots \cap E_n) &\geq P(E_1) + \ldots + P(E_n) - (n-1)
\end{aligned}
$$

as desired.