Statistics 430
HW #3 Solutions

Thanks to Emil Pitkin

October 5, 2011

1 Ch 3 Problem 45

Denote the ith coin by C_i. Because the coins are randomly selected, $P(C_i) = \frac{1}{10}$, and because the coin is fair, $P(H) = P(T) = \frac{1}{2}$. We further assume that $P(H|C_i) = \frac{i}{10}$. Let i range from 1 to 10.

$$\begin{aligned} P(C_5|H) &= \frac{P(H|C_5)P(C_5)}{\sum_{i=1}^{10} P(H|C_i)P(C_i)} \\ &= \frac{\frac{5}{10} * \frac{1}{10}}{\sum_{i=1}^{10} \frac{i}{10} * \frac{1}{10}} \\ &= \frac{5/10}{55/10} \\ &= \boxed{\frac{1}{11} = .091} \end{aligned}$$

2 Ch 3 Problem 55

By the definition on page 71, we know that events A and B are independent if $P(A \cap B) = P(A)P(B)$. Denote membership in the male tribe by M and in the freshman class by Fr. We want $P(M \cap Fr) = P(M)P(Fr)$. Let SG be the number of sophomore girls. For independence, SG must satisfy

$$\begin{aligned} P(M \cap Fr) &= P(M)P(Fr) \\ \frac{4}{4+6+6+SG} &= \frac{4+6}{16+SG}\frac{4+6}{16+SG} \\ \text{rearranging terms} \quad 4(16+SG) &= 100 \\ \boxed{SG} &= \boxed{9} \end{aligned}$$

3 Ch 3 Problem 48

Denote the event "opened drawer reveals silver coin" by S, and the event "both drawers contains silver coins" by B. Note that $B \cap S = B$, since if the opened drawer and both drawers have silver coins, then both drawers have silver coins. We are asked for the probability $P(B|S)$.

$$
\begin{aligned}
P(B|S) &= \frac{P(B\cap S)}{P(S)} \\
&= \frac{P(B)}{P(S)} \\
&= \frac{1/2}{3/4} \\
&= \boxed{\tfrac{2}{3}}
\end{aligned}
$$

4 Ch 3 Problem 60

a

Denote brown by B and blue by b. Because Smith's sister is blue-eyed, she inherited a blue gene from each parent. Since they are both brown-eyed, each must have a Bb pair. Of the brown-eyed offspring, BB, Bb, and bB are equally likely. Since Smith is brown-eyed, his probability of having a blue-eyed gene is $\boxed{\frac{2}{3}}$.

b

Smith's child will have blue eyes if it inherits a blue gene from both parents. Smith's wife will donate her blue gene with probability 1. Brown-eyed Smith will donate a blue gene if his pair is Bb, and then with probability $\frac{1}{2}$. So P(child is blue-eyed) =P(Smith is Bb) $\times$ P(the b is donated|Smith is Bb) $= \frac{2}{3} \times \frac{1}{2} = \boxed{\frac{1}{3}}$.

c

Let us update our probabilities of Smith's genes. Denote by BB the event "Smith ahas two brown genes," and by K_B the event "Smith's first kid has brown eyes". Then

$$
\begin{aligned}
P(BB|K_B) &= \frac{P(K_B|BB)P(BB)}{P(K_B|BB)P(BB) + P(K_B|Bb)P(Bb)} \\
&= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3}} \\
&= \boxed{\tfrac{1}{2}}
\end{aligned}
$$

Denote the event "the second kid will also have brown eyes" by K_{2B}. With our updated probabilities of Smith's genes,

$$
\begin{aligned}
P(K_{2B}) &= P(K_{2B}|BB)P(BB) + P(K_{2B}|Bb)P(Bb) \\
&= 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \\
&= \boxed{\tfrac{3}{4}}
\end{aligned}
$$

5 Ch 3 Problem 73

The denominator for each probability will be 2^5, since a child of either gender can be born for each of the 5 births.

a

Either all boys or all girls. $\frac{1+1}{2^5} = \boxed{\frac{1}{16} = .0625}$

b

The genders are entirely specified: GGBBB. $\frac{1}{2^5} = \boxed{\frac{1}{32} = .03625}$

c

$$\frac{\binom{5}{3}}{2^5} = \boxed{\frac{5}{16} = .3125}$$

d

Only the younger 3 children's genders are unspecified. $\frac{2^3}{2^5} = \boxed{\frac{1}{4} = .25}$

e

The complement of "at least 1 girl" is "all boys." $1 - \frac{1}{2^5} = \boxed{\frac{31}{32} = .96875}$

6 Ch 4 Problem 1

Table 1: Probability distribution of winnings

Balls	Winnings($)	Probability (expression)	Probability (value)
2 white	-2	$\frac{\binom{8}{2}}{\binom{14}{2}}$	$\boxed{\frac{28}{91} = .3077}$
1 white 1 orange	-1	$\frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}}$	$\boxed{\frac{16}{91} = .1758}$
2 orange	0	$\frac{\binom{2}{2}}{\binom{14}{2}}$	$\boxed{\frac{1}{91} = .0110}$
1 white 1 black	1	$\frac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}}$	$\boxed{\frac{32}{91} = .3516}$
1 black 1 orange	2	$\frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}}$	$\boxed{\frac{8}{91} = .0879}$
2 black	4	$\frac{\binom{4}{2}}{\binom{14}{2}}$	$\boxed{\frac{6}{91} = .0659}$

7 Ch 4 Problem 4

The 5 men and 5 women can be ranked in $\binom{10}{5}$ possible ways. If the highest achieving woman attains the ith rank, then no woman occupies the top $i-1$ positions; the 4 remaining women hold 4 of the bottom $10-i$ positions. There are $\binom{10-i}{4}$ ways of selecting 4 rankings for 4 women in the bottom $10-i$ positions. Hence $\boxed{P(X=i) = \frac{\binom{10-i}{4}}{\binom{10}{5}}}$. The following table enumerates the possible values and associated probabilities explicitly.

i	$P(X=i)$
1	$\frac{\binom{9}{4}}{\binom{10}{5}} = \frac{1}{2} = .5$
2	$\frac{\binom{8}{4}}{\binom{10}{5}} = \frac{5}{18} = .278$
3	$\frac{\binom{7}{4}}{\binom{10}{5}} = \frac{5}{36} = .1389$
4	$\frac{\binom{6}{4}}{\binom{10}{5}} = \frac{5}{84} = .0595$
5	$\frac{\binom{5}{4}}{\binom{10}{5}} = \frac{5}{252} = .0198$
6	$\frac{\binom{4}{4}}{\binom{10}{5}} = \frac{1}{252} = .0040$
7-10	0

8 Ch 4 Problem 22

The Red Sox are playing the Yankees Y. Let p denote the probability that the Red Sox win a given game – p could be .6, or .7, or 1, for example. Let N denote the number of games in a series.

a

$P(N=2) = p^2 + (1-p)^2$
$P(N=3) = \binom{2}{1}[p^2(1-p) + (1-p)^2 p] = 2[p(1-p)(p+1-p)] = 2p(1-p)$

$$\begin{aligned} E(N) &= 2P(N=2) + 3P(N=3) \\ &= 2[p^2 + (1-p)^2] + 3 * [2p(1-p)] \\ &= \boxed{2(-p^2 + p + 1)} \end{aligned}$$

Take the first derivative, set is equal to zero, find that the critical point occurs at $\frac{1}{2}$. Confirm that your optimum is indeed a maximum by noticing that the second derivative $(= -4)$ is always negative.

b

$P(N=3) = p^3 + (1-p)^3$
$P(N=4) = \binom{3}{1}[p^3(1-p) + p(1-p)^3]$

$P(N=5) = \binom{4}{2}[p^3(1-p)^2 + p^2(1-p)^3]$

$$\begin{aligned}
E(N) &= 3P(N=3) + 4P(N=4) + 5P(N=5) \\
&= 3[p^3 + (1-p)^3] + 12[p^3(1-p) + p(1-p)^3] + 30[p^3(1-p)^2 + p^2(1-p)^3]] \\
&= 3p^3 + 3 - 9p + 9p^2 - 3p^3 + 12p^3 - 12p^4 + 12p - 36p^2 \\
&+ 36p^3 - 12p^4 + 30p^3 - 60p^4 + 30p^5 + 30p^2 - 90p^3 + 90p^4 - 30p^5 \\
&= \boxed{3(2p^4 - 4p^3 + p^2 + p + 1)} \quad \text{after collecting the 18 terms}
\end{aligned}$$

Substituting $\frac{1}{2}$ for p in $3(8p^3 - 12p^2 + 2p + 1)$ – the first derivative – yields 0, confirming that $\frac{1}{2}$ is an optimum. Substituting $\frac{1}{2}$ for p in $3(24p^2 - 24p + 2)$ – the second derivative – gives -12, confirming that the optimum is indeed a maximum.

9 Ch 4 Problem 23

Because copper vascillates between \$1 and \$4, let's call our commodity C.

a

At the beginning of the week, buy a number of units of C, and sell them at the end of the week. Let \$ denote the amount of money we have at the end of the week.

- With probability $\frac{1}{2}$, $\$ = (1000 - 2C) + 1C$
- With probability $\frac{1}{2}$, $\$ = (1000 - 2C) + 4C$
- Therefore $E[\$] = \frac{1}{2}[1000 - 2C + 1C + 1000 - 2C + 4C] = 1000 + \frac{1}{2}C$.

Subject to the constraint $0 \leq C \leq 500$, we notice that the expected cash on hand is a monotonically increasing function of C. Our strategy is therefore to buy 500 units of the commodity today a

b

At the beginning of the week, buy k units of commodity; at the end of the week buy some more.

- With probability $\frac{1}{2}$, $C = k + \frac{(1000-2k)}{1}$
- With probability $\frac{1}{2}$, $C = k + \frac{(1000-2k)}{4}$
- Therefore $E[C] = \frac{1}{2}[k + (1000 - 2k) + k + \frac{(1000-2k)}{4}] = 625 - \frac{k}{4}$.

We notice that the expected amount of commodity is a monotonically decreasing function of k, the amount purchased at the beginning of the week. Our strategy, then, is to spend all our money on the commodity a week from now.

10 Ch 4 Problem 30

a

Under $\boxed{\text{no}}$ foreseeable circumstances. Let's calculate the probability of winning less than 1 million dollars on our first play – that is, the probability of (possibly miserable) failure.

Denote the event "winnings on nth flip" by T_n, since we lose on the nth flip if we see $n-1$ heads and then a tail.

At least how many consecutive heads must appear for us to win? Since we win 2^n dollars if we win on the nth flip, set $2^n > 1,000,000$. Then $n \log 2 > \log 1,000,000$ and $n > \frac{\log 1,000,000}{\log 2} = 19.93$.
As long as the first tails comes up before the 20th flip, we lose. Therefore

$$\begin{aligned} P(failure) &= \sum_{i=1}^{19} P(T_i) \\ &= \sum_{i=1}^{19} \left(\frac{1}{2}\right)^n \\ &= \sum_{i=0}^{19} \left(\frac{1}{2}\right)^n - 1 \\ &= \frac{1-\left(\frac{1}{2}\right)^{20}}{1-\frac{1}{2}} - 1 \\ &= .999998 \end{aligned}$$

So we have 2 chances in a million of recouping our buy-in.

b

$\boxed{\text{Yes, if you have a well-padded bank account and a lot of time}}$.