Statistics 430
HW #4 Solutions

thanks to Emil Pitkin

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1 Ch 4 Problem 25

a

$P(X = 1) = .6 * (1 - .7) + (1 - .6) * .7 = \boxed{.46}$.

b

- $P(X = 0) = (1 - .6) * (1 - .7) = .12$
- $P(X = 1) = .46$ from part a.
- $P(X = 2) = 1 - .12 - .46 = .42$

$$\begin{aligned} E[X] &= 0P(X = 0) + 1P(X = 1) + 2P(X = 2) \\ &= 0 * .12 + 1 * .46 + 2 * .42 = \boxed{1.3} \end{aligned}$$

2 Ch 4 Problem 28

Let X represent the number of defective items in the sample. Q: under what circumstances will we find i defective items? A: When i of the 3 items in our sample are defective, and the remaining $4 - i$ defectives are buried among the 17 unsampled items. Any 4 of the 20 items could be defective. Therefore

$$\begin{aligned} P(X = i) &= \frac{\binom{3}{i}\binom{17}{4-i}}{\binom{20}{4}} \\ E[X] &= \sum_{i=0}^{3} i \frac{\binom{3}{i}\binom{17}{4-i}}{\binom{20}{4}} \\ &= \frac{0 + 2,040 + 816 + 51}{4845} \\ &= \boxed{\frac{3}{5} = 0.6} \end{aligned}$$

3 Ch 4 Problem 35

a

Let W denote our winnings.

- $P(W = 1.10) = \frac{\binom{5}{2}+\binom{5}{2}}{\binom{10}{2}} = \frac{4}{9}$
- $P(W = -1.00) = \frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}} = \frac{5}{9}$
- $E[W] = 1.1 * \frac{4}{9} - 1 * \frac{5}{9} = \boxed{-\frac{1}{15} = -.067}$ dollars.

b

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] \\ &= \frac{4}{9}[1.10 - (-\frac{1}{15})^2] + \frac{5}{9}[-1.00 - (-\frac{1}{15})^2] \\ &= \boxed{\frac{49}{45} = 1.089} \end{aligned}$$

4 Ch 4 Problem 38

Because $Var[X] = E[X^2] - E[X]^2$, we see that $E[X^2] = Var[X] + E[X]^2$, which in our problem equals $5 + 1^2 = 6$.

a

$$\begin{aligned} E[(2+X)^2] &= E[4 + 4X + X^2] \\ &= E[4] + 4E[X] + E[X^2] \\ &= 4 + 4 * 1 + 6 \\ &= \boxed{14} \end{aligned}$$

b

$$\begin{aligned} Var[4 + 3X] &= Var[3X] \\ &= 3^2 Var[X] \\ &= \boxed{45} \end{aligned}$$

5 Ch 4 Problem 41

If the man were not endowed with ESP-ness, then he would guess each coin's outcome correctly with probability = .5, and his number of correct guesses, X, would be distributed as a $Bin(10, .5)$ random variable. The probability that he would have guessed at least 7 coin flips correctly is:

$$
\begin{aligned}
P(X \geq 7) &= \sum_{k=7}^{10} \binom{10}{k} .5^k .5^{10-k} \\
&= .5^{10}(120 + 45 + 10 + 1) \\
&= \boxed{\tfrac{11}{64} = .1719}
\end{aligned}
$$

6 Ch 4 Problem 49

a

Denote the event "first coin is flipped" by C_1, with C_2 defined similarly. Let X be the number of heads out of 10 tosses.

$$
\begin{aligned}
P(X = 7) &= P(X = 7|C_1)P(C_1) + P(X = 7|C_2)P(C_2) \\
&= [\binom{10}{7} .4^7 .6^3]\frac{1}{2} + [\binom{10}{7} .7^7 .3^3]\frac{1}{2} \\
&= [.0425].5 + [.2668].5 \\
&= \boxed{.1547}
\end{aligned}
$$

b

By conditioning on the outcome of the first flip, we update the probability (now evenly split between coins 1 and 2) that coin 1 is being flipped. Let H_1 denote the event "first flip is heads".

$$
\begin{aligned}
P(C_1|H_1) &= \frac{P(H_1|C_1)P(C_1)}{P(H_1|C_1)P(C_1) + P(H_1|C_2)P(C_2)} \\
&= \frac{.4 * .5}{.4 * .5 + .7 * .5} \\
&= \frac{4}{11}
\end{aligned}
$$

Our updated probabilities are now: $P(C_1) = \frac{4}{11}$ and $P(C_2) = \frac{7}{11}$. It is left to find the probability that 6 of the remaining 9 flips will land on heads – let Y be a $Bin(9, 6)$ random variable.

$$
\begin{aligned}
P(Y = 6) &= P(Y = 6|C_1)P(C_1) + P(Y = 6|C_2)P(C_2) \\
&= [\binom{9}{6} .4^6 .6^3]\frac{4}{11} + [\binom{9}{6} .7^6 .3^3]\frac{7}{11} \\
&= [.0743].3636 + [.2668].6364 \\
&= \boxed{.1968}
\end{aligned}
$$

7 Ch 4 Problem 5*

Claim: $\sum_{i=0}^{\infty} iP(N > i) = \frac{1}{2}E[N^2] - E[N])$
Proof:

$$
\begin{aligned}
\sum_{i=0}^{\infty} iP(N>i) &= \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P(N=k) \\
&= \sum_{i=0}^{k-1} i \sum_{k=1}^{\infty} P(N=k) \\
&= \frac{(k-1)k}{2} \sum_{k=1}^{\infty} P(N=k) \\
&= \frac{1}{2}\Big[k^2 \sum_{k=1}^{\infty} P(N=k) - k \sum_{k=1}^{\infty} P(N=k)\Big] \\
&= \frac{1}{2} E[N^2] - E[N])
\end{aligned}
$$

as we had set out to prove ■

8 Ch 4 Problem 10*

$$
\begin{aligned}
E\left[\frac{1}{X+1}\right] &= \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{1}{k+1} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)!} p^k (1-p)^{n-k} \\
&= \frac{1}{p(n+1)} \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1} (1-p)^{n-k} \\
&= \frac{1}{p(n+1)} \sum_{k=0}^{n} \binom{n+1}{k+1} p^{k+1} (1-p)^{n-k} \\
&= \frac{1}{p(n+1)} \sum_{j=1}^{m} \binom{m}{j} p^{j} (1-p)^{m-j} \quad \text{j=k+1; m=n+1} \\
&= \frac{1}{p(n+1)} \Big[\sum_{j=0}^{m} \binom{m}{j} p^{j} (1-p)^{m-j} - \binom{m}{0} p^{0} (1-p)^{m-0}\Big] \\
&= \frac{1}{p(n+1)} \Big[1-(1-p)^{m}\Big] \\
&= \frac{1}{p(n+1)} \Big[1-(1-p)^{n+1}\Big]
\end{aligned}
$$

as required■ Notice, for example, that when $p = 1$, the expectation of $\frac{1}{X+1}$ is $\frac{1}{n+1}$, as expected.