Statistics 430
HW #5 Solutions

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1 Ch 4 Problem 54 – Poisson paradigm

Because the number of cars speeding along a highway is large, and the probability that a given car is abandoned is small, we can estimate the number of weekly abandoned cars (X) by a Poisson random variable. Here, X $\sim$ Pois(2.2).

a

$$P(X=0) = \frac{e^{-2.2} * 2.2^0}{0!} = \boxed{e^{-2.2} = .111}$$

b

$$P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-2.2} - \frac{e^{-2.2} * 2.2^1}{1!} = \boxed{1 - 3.2e^{-2.2} = .645}$$

2 Ch 4 Problem 57

Let X denote the number of accidents occurring on the highway each day: X $\sim$ Pois(3).

a

$$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2) = 1 - e^{-3} - 3e^{-3} - \frac{e^{-3} * (3)^2}{2!} = \boxed{1 - 8.5e^{-3} = .5768}$$

b

$$\begin{aligned} P(X \geq 3 | X \geq 1) &= \frac{P(X \geq 3 \cap X \geq 1)}{P(X \geq 1)} \\ &= \frac{P(X \geq 3)}{P(X \geq 1)} \\ &= \frac{P(X \geq 3)}{1 - P(X = 0)} \\ &= \frac{.5768}{1 - e^{-3}} \\ &= \boxed{.6070} \end{aligned}$$

3 Ch 4 Problem 60

Let X denote the number of colds Mr. Sniffles contracts during the year: X $\sim$ Pois(λ). We are given that $P(\lambda = 5) = .25$, and $P(\lambda = 3) = .75$. The drug is beneficial if for a given individual, $\lambda = 3$.

$$\begin{aligned} P(\lambda = 3 | X = 2) &= \frac{P(X = 2 | \lambda = 3) P(\lambda = 3)}{P(X = 2 | \lambda = 3) P(\lambda = 3) + P(X = 2 | \lambda = 5) P(\lambda = 5)} \\ &= \frac{.75 * \frac{e^{-3} * 3^2}{2!}}{.75 * \frac{e^{-3} * 3^2}{2!} + .25 * \frac{e^{-5} * 5^2}{2!}} \\ &= \boxed{.8886} \end{aligned}$$

4 Ch 4 Problem 61

We are met with the Poisson paradigm again. We are dealt a large number of poker hands ($n = 1000$), with a small probability of a full house each time ($p = .0014$). The number of full houses (X) can therefore be approximated by a Poisson random variable with mean $np = 1.4$.

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-1.4} - 1.4e^{-1.4} = \boxed{1 - 2.4e^{-1.4} = .4082}$$

5 Ch 4 Problem 71

Denote the event "Smith wins his ith bet" by W_i. Then $P(W_i)$ *is* $\frac{12}{38}$. We assume that the roulette spins are independent.

a

$P(\bigcup_{i=1}^{5} W_i^c) = \prod_{i=1}^{5} P(W_i)^c = \boxed{\left(\frac{26}{38}\right)^5 = .1500}$.

b

$$\boxed{\left(\frac{26}{38}\right)^3\left(\frac{12}{38}\right) = .1012}$$

6 Ch 4 Problem 74

a

The number of potential interviewees who consent to the interview (X) is a Binomial random variable, with $n = 5$ and $p = \frac{2}{3}$. Q: What is the probability that each of the 5 people consents to the interview?

A: $P(X = 5) = \binom{5}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0 = \boxed{\left(\frac{2}{3}\right)^5 = \frac{32}{243} = .1317}$.

b

Now $X \sim \text{Binom}(8, \frac{2}{3})$.

$$\begin{aligned} P(X \geq 5) &= \boxed{\textstyle\sum_{k=5}^{8}\binom{8}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{8-k}} \\ &= \boxed{.7414} \end{aligned}$$

c

We are asked for the probability that the 6th potential interviewee will be the 5th to consent.

$$\begin{aligned} P(X = 6) &= \binom{5}{4}\left(\frac{2}{3}\right)^5\frac{1}{3} \\ &= \boxed{\frac{160}{729} = .2731} \end{aligned}$$

d

$$\begin{aligned} P(X = 7) &= \binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2 \\ &= \boxed{\frac{160}{729} = .2731} \end{aligned}$$

7 Ch 4 Problem 76

The pipe-smoking mathematician carries N_1 matches in his left pocket, and N_2 in his right. When he discovers that one of the boxes is empty, let X be the number of matches left in the other box.

We calculate $P(X = k)$. First consider the case when he discovers that the left box is empty. He must have reached into the left box $N_1 + 1$ times (N times he retrieved a match;

the next time he discovered it was empty). Because, by assumption, k matches remain in the right-hand box, he must have reached into it $N_2 - k$ times. He therefore reached for matches a total of $N_1 + N_2 + 1 - k$ times.

The last time he reached, he must have reached into the left-hand pocket. Of the first $N_1 + N_2 - k$ reaches, he reached into the left-hand pocket N_1 times, making for a total of $\binom{N_1+N_2-k}{N_1}$ possible sequences of reaches. Since he chooses the right or left pocket with probability $\frac{1}{2}$ each time, the probability of any given sequence of $N_1 + N_2 + 1 - k$ reaches occurs with probability $\left(\frac{1}{2}\right)^{2N+1-k}$.

The probability of k matches remaining in the right hand pocket is therefore $\binom{N_1+N_2-k}{N_1}\left(\frac{1}{2}\right)^{N_1+N_2-k}$ By symmetry, the probability of k matches remaining in the left hand pocket when the right pocket is found empty is $\binom{N_1+N_2-k}{N_2}\left(\frac{1}{2}\right)^{N_1+N_2-k+1}$. Adding, we find that

$$\boxed{P(X = k) = \binom{N_1+N_2-k}{N_1}\left(\frac{1}{2}\right)^{N_1+N_2-k+1} + \binom{N_1+N_2-k}{N_2}\left(\frac{1}{2}\right)^{N_1+N_2-k+1}}$$

8 Ch 4 Problem 78

Because we keep selecting (that is, failing) until exactly two of the balls we draw are white, the number of selections, X, is a geometric random variable, with parameter $p = \frac{\binom{4}{2}^2}{\binom{8}{4}} = \frac{18}{35}$.

$$\begin{aligned} P(X = n) &= (1-p)^{n-1}p \\ &= (1 - \frac{18}{35})^{n-1} * \frac{18}{35} \\ &= \boxed{\frac{(17)^{n-1}18}{(35)^n}} \end{aligned}$$

9 Ch 4 Problem 79

X is a hypergeometric random variable.

a

$$\begin{aligned} P(X = 0) &= \frac{\binom{6}{0}\binom{94}{10}}{\binom{100}{10}} \\ &= \boxed{.5223} \end{aligned}$$

b

$$\begin{aligned} P(X > 2) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= 1 - .5223 - \frac{\binom{6}{1}\binom{94}{9}}{\binom{100}{10}} - \frac{\binom{6}{2}\binom{94}{8}}{\binom{100}{10}} \\ &= \boxed{.0126} \end{aligned}$$

10 Ch 4 Problem 80

a

A payoff is "fair" when the expected winnings are 0. Without loss of generality, assume that the gambler bets \$1. The player wins his bet with probability $\frac{20*19}{80*79} = \frac{19}{316}$, and loses his dollar with probability $\frac{297}{316}$. Let X be the gambler's expected winnings. In order for $E[X] = 0$, the payoff F must be such that

$$\begin{aligned} \frac{297}{316} * (-1) + \frac{19}{316} F &= 0 \\ F &= \boxed{\frac{297}{19} = 15.63} \end{aligned}$$

Unsurprisingly, the casino f***s over the gambler.

b

$P_{n,k}$ is a hypergeometric random variable. The casino selects the 20 numbers in any of $\binom{80}{20}$ possible ways. We choose n numbers, of which k must be winning numbers ($\binom{20}{k}$ such numbers), and the remaining $20 - k$ winning numbers must lie among the $80 - n$ numbers we did not choose ($\binom{80-n}{20-k}$ ways).

$$\boxed{P_{n,k} = \frac{\binom{20}{k}\binom{80-n}{20-k}}{\binom{80}{20}}}$$

c

Let F be the payoff.

$$\begin{aligned} E[F] &= -1 * \frac{\binom{10}{0}\binom{70}{20-0}}{\binom{80}{20}} + -1 * \frac{\binom{10}{1}\binom{70}{20-1}}{\binom{80}{20}} + -1 * \frac{\binom{10}{2}\binom{70}{20-2}}{\binom{80}{20}} + -1 * \frac{\binom{10}{3}\binom{70}{20-3}}{\binom{80}{20}} \\ &+ -1 * \frac{\binom{10}{4}\binom{70}{20-4}}{\binom{80}{20}} + 1 * \frac{\binom{10}{5}\binom{70}{20-5}}{\binom{80}{20}} + 17 * \frac{\binom{10}{6}\binom{70}{20-6}}{\binom{80}{20}} + 179 * \frac{\binom{10}{7}\binom{70}{20-7}}{\binom{80}{20}} \\ &+ 1299 * \frac{\binom{10}{8}\binom{70}{20-8}}{\binom{80}{20}} + 2599 * \frac{\binom{10}{9}\binom{70}{20-9}}{\binom{80}{20}} + 24999 * \frac{\binom{10}{10}\binom{70}{20-10}}{\binom{80}{20}} \\ &= \boxed{-.206} \end{aligned}$$

11 Ch 4 Problem 84

a

Define the following indicator random variables:

$$I_i = \left\{ \begin{array}{ll} 1 & \text{Box i is empty} \\ 0 & \text{otherwise} \end{array} \right\}$$

The number of empty boxes is given by $\sum_{i=1}^{5} I_i$. What are we interested in? The expected number of boxes with no balls lodged inside, which is given by

$$
\begin{aligned}
E[\sum_{i=1}^{5} I_i] &= \sum_{i=1}^{5} E[I_i] \\
&= \sum_{i=1}^{5} P(I_i = 1) \\
&= \boxed{\sum_{i=1}^{5} (1-p_i)^{10}}
\end{aligned}
$$

b

We proceed analagously. Define the following indicator random variables:

$$
I_i = \left\{ \begin{array}{ll} 1 & \text{Box i has one ball inside} \\ 0 & \text{otherwise} \end{array} \right\}
$$

The number of boxes with a single ball is given by $\sum_{i=1}^{5} I_i$. What are we interested in? The expected number of boxes with exactly one ball lodged inside, which is given by

$$
\begin{aligned}
E[\sum_{i=1}^{5} I_i] &= \sum_{i=1}^{5} E[I_i] \\
&= \sum_{i=1}^{5} P(I_i = 1) \\
&= \boxed{\sum_{i=1}^{5} \binom{10}{1} (p_i)^1 (1-p_i)^9}
\end{aligned}
$$

12 Ch 4 Theoretical Exercise 18*

We are asked to find the value of λ that maximizes $P(X = k)$, where X is a Pois(λ) random variable.

$$
P(X = k) = \frac{e^{-\lambda} * \lambda^k}{k!}
$$

Take the derivative with respect to λ, set the expression equal to 0, and solve for λ:

$$
\begin{aligned}
\frac{1}{k!}\left(e^{-\lambda} * k\lambda^{k-1} - e^{-\lambda}\lambda^k\right) &= 0 \\
\underbrace{\left[\frac{1}{k!}e^{-\lambda} * \lambda^{k-1}\right]}_{\text{kill it}} (k - \lambda) &= 0 \\
&\Rightarrow \boxed{\lambda^* = k}
\end{aligned}
$$

13 Ch 4 Theoretical Exercise 19*

Claim:

$$
E[X^n] = \lambda E[(X+1)^{n-1}]
$$

Proof:

$$\begin{aligned}
\lambda E[(X+1)^{n-1}] &= \lambda \sum_{k=0}^{\infty} \frac{(k+1)^{n-1} e^{-\lambda} \lambda^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(k+1)^{n} e^{-\lambda} \lambda^{k+1}}{(k+1)!} \\
&= \sum_{j=1}^{\infty} \frac{(j)^{n} e^{-\lambda} \lambda^{j}}{(j)!} \\
&= \sum_{j=0}^{\infty} \frac{(j)^{n} e^{-\lambda} \lambda^{j}}{(j)!} \\
&= E[X^n] \quad \blacksquare
\end{aligned}$$

$$\begin{aligned}
E[X^3] &= \lambda E[(X+1)^2] \\
&= \lambda(E[X^2 + 2E[X] + 1]) \\
&= \lambda(\lambda E[X+1] + 2E[X] + 1) \\
&= \lambda(\lambda^2 + \lambda + 2\lambda + 1) \\
&= \boxed{\lambda^3 + 3\lambda^2 + \lambda}
\end{aligned}$$