Statistics 430
HW #6 Solutions

Hyunseung Kang, Emil Pitkin, and Yun Zhang

October 31, 2011

1 Ch 5 Problem 3

$\boxed{\text{Neither}}$ is a density function. Both $2x - x^3$ and $2x - x^2$ attain positive and negative values in the interval $0 < x < \frac{5}{2}$, so no matter the value of C, $f(x)$ is negative somewhere in the interval. In contrast, density functions are, by definition, non-negative.

2 Ch 5 Problem 5

When the tank's capacity is the 99th percentile of sales volume, the probability of the supply's being exhausted in a given week is .01. Let C denote the tank's capacity. We wish for C to satisfy

$$\begin{aligned} 5\int_0^C (1-x)^4 dx &= .99 \\ -(1-x)^5\Big|_0^C &= .99 \\ (1-C)^5 &= .01 \\ &\Rightarrow \boxed{C = 1 - \sqrt[5]{.01} = .602} \end{aligned}$$

The tank's capacity should therefore be 602 gallons.

3 Ch 5 Problem 7

We will solve two simultaneous equations. We are equipped with two pieces of knowledge:
1) $E[X] = \frac{3}{5}$ and
2) the density must integrate to 1.

Expectation equation:

$$\begin{aligned} \int_0^1 x(a + bx^2)dx &= \frac{3}{5} \\ (\frac{a}{2}x^2 + \frac{b}{4}x^4)\Big|_0^1 &= \frac{3}{5} \\ \frac{a}{2} + \frac{b}{4} &= \frac{3}{5} \end{aligned}$$

Density equation:

$$\begin{aligned} \int_0^1 a + bx^2 dx &= 1 \\ ax + \frac{b}{3}x^3 \Big|_0^1 &= 1 \\ a + \frac{b}{3} &= 1 \end{aligned}$$

Solving the two equations for a and b yields $\boxed{a = \frac{3}{5}}$ and $\boxed{b = \frac{6}{5}}$

4 Ch 5 Problem 8*

Integrate by parts:

$$\begin{aligned} E[X] &= \int_0^\infty x^2 e^{-x} dx \\ &= -x^2 e^{-x}\Big|_0^1 - 2\int_0^1 xe^{-x}dx \\ &= -\frac{1}{e} - 2\left[-xe^{-x}\Big|_0^1 - \int_0^1 e^{-x}dx\right] \\ &= -\frac{1}{e} - 2\left[-\frac{1}{e} + e^{-x}\Big|_0^1\right] \\ &= -\frac{1}{e} - 2\left[-\frac{1}{e} + \left(\frac{1}{e} - 1\right)\right] \\ &= \boxed{\text{2 hours}} \end{aligned}$$

5 Ch 5 Problem 11

Let X be the location of the chosen point. Then $X \sim Unif[0, L]$. We can assume without loss of generality that L equals 1.

$$\begin{aligned} P\left(\frac{X}{1-X} < \frac{1}{4}\right) + P\left(\frac{X}{1-X} > 4\right) &= P(4X < 1 - X) + P(X > 4 - 4X) \\ &= P\left(X < \frac{1}{5}\right) + \left(X > \frac{4}{5}\right) \\ &= .2 + .2 = \boxed{.4} \end{aligned}$$

6 Ch 5 Problem 12

The more efficient placement would minimize the expected distance between breakdown point and bus service station. Let B denote the breakdown point. Under the present arrangement, the minimum distance M between B and a service station is given by

$$
\begin{aligned}
M &= \begin{cases} B & 0 < B < 25 \\ 50 - B & 25 < B < 50 \\ B - 50 & 50 < B < 75 \\ 100 - B & 75 < B < 100 \end{cases} \\
E[M] &= \int_0^{25} B * \frac{1}{100} dB + \int_{25}^{50} (50 - B) * \frac{1}{100} dB \\
&\quad + \int_{50}^{75} (B - 50) * \frac{1}{100} dB + \int_{75}^{100} (100 - B) * \frac{1}{100} dB \\
&= \frac{1}{100}\left(\frac{B^2}{2}\Big|_0^{25} + \left[50B - \frac{B^2}{2}\right]\Big|_{25}^{50} + \left[\frac{B^2}{2} - 50B\right]\Big|_{50}^{75} + \left[100B - \frac{B^2}{2}\right]\Big|_{75}^{100} \right) \\
&= \frac{1}{100}\left[\frac{25^2 - 0^2}{2} + \left(50 * 25 - \frac{50^2 - 25^2}{2}\right) \right] \\
&\quad + \frac{1}{100}\left[\left(\frac{75^2 - 50^2}{2} - 50 * 25\right) + \left(50 * 25 - \frac{100^2 - 75^2}{2}\right) \right] \\
&= \boxed{12.5 \text{ miles}}
\end{aligned}
$$

We could have avoided the mess and arrived at the same happy ending by arguing by symmetry.

Under the alternate arrangement, the minimum distance M between B and a service station is given by

$$
M = \begin{cases} B & 0 < B < 12.5 \\ 25 - B & 12.5 < B < 25 \\ B - 25 & 25 < B < 37.5 \\ 50 - B & 37.5 < B < 50 \\ B - 50 & 50 < B < 62.5 \\ 75 - B & 62.5 < B < 75 \\ B - 75 & 75 < B < 87.5 \\ 100 - B & 37.5 < B < 50 \end{cases}
$$

With analagous calculations and reasoning, we find that the expected distance between the breakdown point and nearest rest stop is $\boxed{6.25 \text{ miles}}$. Indeed, the alternate arrangement is more efficient.

7 Ch 5 Problem 13

Let A be the number of minutes past 10 o'clock when the bus arrives. $A \sim Unif[0, 30]$. Recall that the CDF of a $Unif[a, b]$ random variable is given by $F(x) = \frac{x-a}{b-a}$.

a

$$
\begin{aligned}
P(A > 10) &= 1 - F(10) \\
&= 1 - \frac{10 - 0}{30 - 0} = \boxed{\frac{2}{3}}
\end{aligned}
$$

b

$$
\begin{aligned}
P(A > 25 | A > 15) &= \frac{P(A > 25 \cap A > 10)}{P(A > 15)} \\
&= \frac{P(A > 25)}{P(A > 15)} \\
&= \tfrac{1}{6} / \tfrac{1}{2} \\
&= \boxed{\tfrac{1}{3}}
\end{aligned}
$$

8 Ch5 Problem 14*

If $X \sim U(0,1)$, then $f(x) = 1$ for $0 < x < 1$ and $F(x) = x$ for $0 < x < 1$. Since $0 < X < 1$, so $0 < X^n < 1$. By using Proposition 2.1,

$$
\begin{aligned}
E(X^n) &= \int_0^1 P(X^n > t) dt \\
&= \int_0^1 P(X > t^{1/n}) dt \\
&= \int_0^1 1 - P(X \leq t^{1/n}) dt \\
&= \int_0^1 1 - F(t^{1/n}) dt \\
&= \int_0^1 1 - t^{1/n} dt \\
&= \left[t - \frac{t^{\frac{1}{n}+1}}{\frac{1}{n}+1} \right]_0^1 \\
&= 1 - \frac{n}{n+1} \\
&= \frac{1}{n+1}.
\end{aligned}
$$

By using the definition of expectation, we first have to derive the pdf for $f(x^n)$. To do this, we find the cumulative distribution function and take the derivative (aka $F_{X^n}(x) = \int_0^x f(x)dx \Rightarrow \frac{d}{dx} F_{X^n}(x) = f(x)$, by the Fundamental Theorem of Calculus)

$$
F_{X^n}(x) = P(X^n < x) = P(X < x^{1/n}) = x^{1/n} \Rightarrow \frac{1}{n} x^{1/n-1}
$$

Then, our expectation is:

$$\begin{aligned}
E(X^n) &= \int_0^1 x \frac{1}{n} x^{1/n-1} dx \\
&= \int_0^1 \frac{1}{n} x^{\frac{1}{n}} dx \\
&= \frac{1}{n} \left[\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} \right]_0^1 \\
&= \frac{1}{n} \left[\frac{1}{\frac{1}{n}+1} \right] \\
&= \frac{1}{n+1}.
\end{aligned}$$

9 Ch 5 Problem 15

a

$$\begin{aligned}
P(X > 5) &= P\left(\frac{X-\mu}{\sigma} > \frac{5-10}{6}\right) \\
&= P(Z > -.83) \\
&= \boxed{.798}
\end{aligned}$$

b

$$\begin{aligned}
P(4 < X < 16) &= P(Z < \frac{16-10}{6}) - P(Z < \frac{4-10}{6}) \\
&= P(Z < 1) - P(Z < -1) \\
&= \boxed{.683}
\end{aligned}$$

c

$$\begin{aligned}
P(X < 8) &= P(Z < \frac{8-10}{6}) \\
&= P(Z < -.33) \\
&= \boxed{.369}
\end{aligned}$$

d

$$\begin{aligned}
P(X < 20) &= P(Z < \frac{20-10}{6}) \\
&= P(Z < 1.67) \\
&= \boxed{.952}
\end{aligned}$$

e

$$\begin{aligned} P(X > 16) &= P(Z > \frac{16-10}{6}) \\ &= P(Z > 1) \\ &= \boxed{.159} \end{aligned}$$

10 Ch 5 Problem 17

Let D denote the distance from the shot to the target. We are given that $D \sim Unif[0, 10]$. The winnings W are distributed as follows:

$$\begin{aligned} W &= \begin{cases} 10 & 0 < D < 1 \\ 5 & 1 < D < 3 \\ 3 & 3 < D < 5 \\ 0 & 5 < D < 10 \end{cases} \\ E[W] &= \sum_{w_i} w_i p(w_i) \\ &= 10 * \frac{1}{10} + 5 * \frac{2}{10} + 3 * \frac{2}{5} + 0 * \frac{1}{2} \\ &= \boxed{2.6} \end{aligned}$$

11 Ch5 Problem 18*

Let X be a normal random variable with mean 5 and variance σ^2. $X \sim N(5, \sigma^2)$. If $0.2 = P(X > 9)$, then

$$\begin{aligned} 0.8 &= P(X \leq 9) \\ &= P\left(\frac{X-5}{\sigma} \leq \frac{9-5}{\sigma}\right), \text{ where } \frac{X-5}{\sigma} = Z \sim N(0,1) \\ &= \Phi\left(\frac{4}{\sigma}\right) \end{aligned}$$

From a standard normal table, we have $\Phi^{-1}(0.8) = 0.842$. Thus

$$\frac{4}{\sigma} = 0.842 \Rightarrow \sigma = 4.75.$$

Hence $Var(X) = \sigma^2 = 22.56$.

12 Ch5 Theoretical Problem 2*

In order to show that

$$E[Y] = \int_0^\infty P(Y > y)dy - \int_0^\infty P(Y < -y)dy,$$

it is equivalent to show that

$$\begin{aligned}
\int_0^\infty P(Y < -y)dy &= -\int_{-\infty}^0 x f_Y(x)dx && (1)\\
\int_0^\infty P(Y > y)dy &= \int_0^\infty x f_Y(x)dx. && (2)
\end{aligned}$$

Now we are showing (1),

$$\begin{aligned}
LHS &= \int_{y=0}^\infty \left(\int_{x=-\infty}^{-y} f_Y(x)dx \right) dy\\
&= \int_{x=-\infty}^0 \left(\int_{y=0}^{-x} 1dy \right) f_Y(x)dx\\
&= \int_{x=-\infty}^0 -x f_Y(x)dx\\
&= RHS.
\end{aligned}$$

And showing (2),

$$\begin{aligned}
LHS &= \int_{y=0}^\infty \left(\int_{x=y}^{\infty} f_Y(x)dx \right) dy\\
&= \int_{x=0}^\infty \left(\int_{y=0}^{x} 1dy \right) f_Y(x)dx\\
&= \int_{x=0}^\infty x f_Y(x)dx\\
&= RHS.
\end{aligned}$$

Combining (1) and (2), we have

$$\begin{aligned}
E[Y] &= \int_{-\infty}^\infty y f_Y(y)dy\\
&= \int_{-\infty}^0 x f_Y(x)dx + \int_0^\infty x f_Y(x)dx, \text{ where } x, y \text{ are dummy}\\
&= (2) - (1).
\end{aligned}$$

13 Ch5 Theoretical Problem 5*

For a nonnegative random variable X, (X^n nonnegative) we have

$$E[X^n] = \int_{t=0}^\infty P(X^n > t)dt.$$

Using change of variables $t = x^n$, $\{X^n > x^n\} \Leftrightarrow \{X > x\}$ and $\frac{dt}{dx} = nx^{n-1}$,

$$
\begin{aligned}
E[X^n] &= \int_{x=0}^{\infty} P(X > x)\frac{dt}{dx}dx \\
&= \int_{x=0}^{\infty} P(X > x)nx^{n-1}dx, \text{ as required.}
\end{aligned}
$$

14 Ch5 Theoretical Problem 8

15 Ch5 Theoretical Problem 13

We derive all of these medians using the definition of the median (i.e. m is a median of X if $F(m) = \frac{1}{2}$), rather than intuition (because it's obvious, otherwise)

15.1 a

15.2 b

15.3 c

16 Ch5 Theoretical Problem 21

Using $x \to \sqrt{2x}$ change of variables, we get the following

$$
\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-x}x^{-1/2}dx = \int_0^{\infty} e^{-\frac{1}{2}y^2}\left(\frac{1}{2}y^2\right)^{-1/2} ydy = \int_0^{\infty} \sqrt{2}e^{-\frac{1}{2}y^2}dy = \sqrt{\pi}
$$

where the last equality uses $\int_0^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy = \frac{1}{2}$