Statistics 430
HW #8 Solutions

Emil Pitkin

November 17, 2010

1 Ch 6 Problem 1

a

For $1 \le i \le 6$ and $2 \le j \le 12$,

$$p(i,j) = \begin{cases} \frac{1}{36} & i = \frac{j}{2} \\ \frac{2}{36} & i < j < 2i \\ 0 & \text{otherwise} \end{cases}$$

b

For $1 \le i, j \le 6$,

$$p(i,j) = \begin{cases} \frac{i}{36} & i = j \\ \frac{1}{36} & i < j \\ 0 & \text{otherwise} \end{cases}$$

c

For $1 \le i, j \le 6$,

$$p(i,j) = \begin{cases} \frac{1}{36} & i = j \\ \frac{2}{36} & i < j \\ 0 & \text{otherwise} \end{cases}$$

2 Ch 6 Problem 2

a

Let $p(i,j) = P(X_1 = i, X_2 = j)$.

$$\begin{cases} p(1,1) = \frac{5}{13} * \frac{4}{12} = \boxed{\frac{5}{39} = .128} \\ p(1,0) = \frac{5}{13} * \frac{8}{12} = \boxed{\frac{10}{39} = .256} \\ p(0,1) = \frac{8}{13} * \frac{5}{12} = \boxed{\frac{10}{39} = .256} \\ p(0,0) = \frac{8}{13} * \frac{7}{12} = \boxed{\frac{15}{39} = .359} \end{cases}$$

b

Let $p(i,j,k) = P(X_1 = i, X_2 = j, X_3 = k)$.

$$\begin{cases} p(1,1,1) = \frac{5}{13} * \frac{4}{12} * \frac{3}{11} = \boxed{\frac{15}{429} = .035} \\ p(1,1,0) = p(1,0,1) = p(0,1,1) = \frac{5}{13} * \frac{4}{12} * \frac{8}{11} = \boxed{\frac{40}{429} = .093} \\ p(0,0,1) = p(0,1,0) = p(1,0,0) = \frac{8}{13} * \frac{7}{12} * \frac{5}{11} = \boxed{\frac{70}{429} = .163} \\ p(0,0,0) = \frac{8}{13} * \frac{7}{12} * \frac{6}{11} = \boxed{\frac{84}{429} = .196} \end{cases}$$

3 Ch 6 Problem 7

The number of failures preceding the first success follows a geometric distribution with parameter p. Accordingly, $P(X_1 = i) = (1-p)^i p$. The number of failures between the first two success is likewise a geometric random variable: $P(X_2 = j) = (1-p)^j p$. Because X_1 and X_2 are independent,

$$\begin{aligned} P(X_1 = i, X_2 = j) &= P(X_1 = i)P(X_2 = j) \\ &= (1-p)^i p (1-p)^j p \\ &= p^2 (1-p)^{i+j} \end{aligned}$$

4 Ch 6 Problem 9

a

$$\begin{aligned} \int_0^2 \int_0^1 f(x,y)\, dx\, dy &= \frac{6}{7} \int_0^2 \int_0^1 x^2 + \frac{xy}{2}\, dx\, dy \\ &= \frac{6}{7} \int_0^2 \frac{1}{3} + \frac{y}{4}\, dy \quad \text{(note for part f: } f_Y(y) = \frac{1}{3} + \frac{y}{4}) \\ &= \frac{6}{7}\left(\frac{2}{3} + \frac{1}{2}\right) = \boxed{1} \end{aligned}$$

b

$$\begin{aligned} f_X(x) &= \int_0^2 f(x,y)\, dy \\ &= \frac{6}{7} \int_0^2 x^2 + \frac{xy}{2}\, dy \\ &= \frac{6}{7}\left(x^2 y + \frac{xy^2}{4}\Big|_0^2\right) \\ &= \boxed{\frac{6}{7}\left(2x^2 + x\right)} \end{aligned}$$

c

$$\begin{aligned}
P(X > Y) &= \frac{6}{7}\int_0^1 \int_0^x x^2 + \frac{xy}{2}\ dy\ dx \\
&= \frac{6}{7}\int_0^1 \left(x^2 y + \frac{xy^2}{4}\right)\Big|_0^x\ dx \\
&= \frac{6}{7}\int_0^1 x^3 + \frac{x^3}{4}\ dx \\
&= \frac{6}{7}\left(\frac{x^4}{4} + \frac{x^4}{16}\right)\Big|_0^1 \\
&= \boxed{\tfrac{15}{56} = .268}
\end{aligned}$$

d

$$\begin{aligned}
P\left(Y > \frac{1}{2}\Big| X < \frac{1}{2}\right) &= \frac{P\left(Y > \frac{1}{2}, X < \frac{1}{2}\right)}{P\left(X < \frac{1}{2}\right)} \\
&= \frac{\int_{\frac{1}{2}}^2 \int_0^{\frac{1}{2}} f(x,y)\ dx\ dy}{\int_0^{\frac{1}{2}} f_X(x)\ dx} \\
&= \frac{\frac{6}{7}\int_{\frac{1}{2}}^2 \left(\frac{x^3}{3} + \frac{x^2 y}{4}\right)\Big|_0^{\frac{1}{2}}\ dy}{\frac{6}{7}\left(\frac{2x^3}{3} + \frac{x^2}{2}\right)\Big|_0^{\frac{1}{2}}} \\
&= \frac{\int_{\frac{1}{2}}^2 \frac{1}{24} + \frac{y}{16}\ dy}{\frac{1}{12} + \frac{1}{8}} \\
&= \frac{\frac{23}{128}}{\frac{5}{24}} \\
&= \boxed{\tfrac{69}{80} = .8625}
\end{aligned}$$

e

$$\begin{aligned}
E[X] &= \int_0^1 x f_X(x)\ dx \\
&= \frac{6}{7}\int_0^1 x\left(2x^2 + x\right)\ dx \\
&= \frac{6}{7}\left(\frac{x^4}{2} + \frac{x^3}{3}\right)\Big|_0^1 \\
&= \boxed{\tfrac{5}{7} = .714}
\end{aligned}$$

f

$$
\begin{aligned}
E[Y] &= \int_0^2 y f_Y(y)\ dy \\
&= \frac{6}{7}\int_0^2 y\left(\frac{1}{3}+\frac{y}{4}\right)\ dx \\
&= \frac{6}{7}\left(\frac{y^2}{6}+\frac{y^3}{12}\right)\Big|_0^2 \\
&= \boxed{\tfrac{8}{7} = 1.143}
\end{aligned}
$$

5 Ch 6 Problem 14

Let X denote the location of the accident, and let Y denote the location of the wambulance at the time of the accident. We are given that $X \sim \text{Unif}(0, L)$ and $Y \sim \text{Unif}(0, L)$. We are asked for the distribution function of $|X - Y|$, the distance between the accident and the ambulance.

Here is our strategy:

- Define a random variable $Z = -Y$, so that we can write $X - Y$ as the sum $X + Z$:

  Let $Z = -Y \sim \text{Unif}(-L, 0)$. Z has density function $\frac{1}{L}$ on $(-L, 0)$.

- Notice that $X + Z$ is a symmetric distribution, so the density of $|X + Z|$ is twice the density of $X + Z$ when $X + Z$ is between 0 and L:

$$
f_{|X+Z|}(a) = 2 * f_{X+Z}(a) \quad 0 < \text{a} < \text{L}
$$

- Write the density of $X + Z$ using the convolution formula:

$$
f_{X+Z}(a) = \int f_X(x) f_Z(a-x) dx
$$

- Find the limits of integration:

  As above, we let $0 < a < L$. $f_Z(a - x)$ is nonzero when $-L < a - x < 0$, that is, when $a < x < L + a$. Since x is constrained to lie between 0 and L, we need $a < x < L$.

$$
\begin{aligned}
f_{X+Z}(a) &= \int_a^L f_X(x) f_Z(a-x) dx \\
&= \frac{1}{L}\int_a^L f_Z(a-x) dx \\
&= \frac{1}{L^2}\int_a^L dx \\
&= \frac{L-a}{L^2}
\end{aligned}
$$

- Combine:

$$
\begin{aligned}
f_{|X-Y|}(a) &= 2f_{X-Y}(a) \quad 0 < \text{a} < \text{L} \\
&= \boxed{\tfrac{2(L-a)}{L^2}}
\end{aligned}
$$

6 Ch 6 Problem 16

a

$$\boxed{A = \bigcup_{i=1}^{n} A_i}$$

b

Indeed, the A_i are mutually exclusive.

c

Let us first find $P(A_i)$. The event A_i occurs when all n points lie in a semicircle beginning at the point P_i. Said otherwise, n-1 points lie within 180^0 of P_i clockwise. Since the number of degress between P_i and each of the other P_j is uniformly distributed between 0 and 360, $P(P_j < 180) = \frac{1}{2}$. Because the locations of the points are independent, $P(A_i) = \left(\frac{1}{2}\right)^{n-1}$

$$
\begin{aligned}
P(A) &= P\left(\bigcup_{i=1}^{n} A_i\right) \\
&= \sum_{i=1}^{n} P(A_i) \\
&= \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{n-1} \\
&= \boxed{n\left(\tfrac{1}{2}\right)^{n-1}}
\end{aligned}
$$

7 Ch 6 Problem 17

Argue by symmetry. The probability that X_2 lies between X_1 and $X_3(\equiv p_2)$[1] is equal to the probability that X_1 lies between X_2 and $X_3(\equiv p_1)$ and that X_3 lies between X_1 and $X_2(\equiv p_3)$. One, and only one of these events must occur . So $p_1 + p_2 + p_3 = 1$ and $p_1 = p_2 = p_3$. Therefore $p_2 = \boxed{\frac{1}{3}}$.

[1] The triple bar means “defined as.”

8 Ch 6 Problem 18

Let us mildly reformulate the question: let X be the distance between the left point and the midpoint, and let Y be the distance between the right point and the midpoint. We can, without loss of generality, let L = 2, so that X and Y are both $Unif(0,1)$ random variables. Since the ratio of $\frac{L}{3}$ to $\frac{L}{2}$ is $\frac{2}{3}$, we want to find $P(X+Y > \frac{2}{3})$.

We know, from section 6.3.1, that $X+Y$ has the triangular distribution, with

$$f_{X+Y}(a) = \begin{cases} a & 0 \le a \le 1 \\ 2-a & 1 < a < 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\begin{aligned} P\left(X+Y > \frac{2}{3}\right) &= 1 - P\left(X+Y < \frac{2}{3}\right) \\ &= 1 - \int_0^{\frac{2}{3}} a \, da \\ &= 1 - \frac{4/9}{2} \\ &= \boxed{\frac{7}{9}} \end{aligned}$$

9 Ch 6 Problem 19

To verify that f(x,y) is a density function, see whether it integrates to 1. Indeed,

$$\begin{aligned} \int_0^1 \int_0^x \frac{1}{x} \, dy \, dx &= \int_0^1 \frac{y}{x}\Big|_0^x dx \\ &= \int_0^1 1 \, dx \\ &= 1 \end{aligned}$$

a

$$\begin{aligned} f_Y(y) &= \int_y^1 \frac{1}{x} \, dx \\ &= \ln x \Big|_y^1 \\ &= \ln 0 - \ln y \\ &= \boxed{-\ln y} \end{aligned}$$

b

$$\begin{aligned} f_X(x) &= \int_0^x \frac{1}{x}\ dy \\ &= \frac{y}{x}\bigg|_0^x \\ &= \frac{x}{x} - \frac{0}{x} \\ &= \boxed{1} \end{aligned}$$

c

From part b, X is uniform between 0 and 1, so $\boxed{E[X] = \frac{1}{2}}$.

d

$$\begin{aligned} E[Y] &= \int_0^1 y f_Y(y)\ dy \\ &= \int_0^1 -y * \ln y\ dy \end{aligned}$$

What time is it? Time to integrate by parts (and to remember that $\int \ln y = y \ln y - y$):

$$\begin{aligned} E[Y] = -\int_0^1 y * \ln y\ dy &= -\left[y(y\ln y - y)\Big|_0^1 - \int_0^1 y \ln y - y\ dy \right] \\ 2\int_0^1 y * \ln y\ dy &= y(y \ln y - y)\Big|_0^1 - \int_0^1 -y\ dy = -1 + \frac{1}{2} = -\frac{1}{2} \\ \int_0^1 -y * \ln y\ dy &= \boxed{\tfrac{1}{4}} \end{aligned}$$