Statistics 430
HW #8 Solutions

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1 Ch 6 Problem 23

$$
\begin{aligned}
f_X(x) &= \int_0^1 f(x,y)\ dy \\
&= \int_0^1 12xy(1-x)\ dy \\
&= 6x(1-x) \quad 0 < x < 1 \\
f_Y(y) &= \int_0^1 f(x,y)\ dx \\
&= 2y \quad 0 < y < 1
\end{aligned}
$$

a

Because $f(x,y) = f_X(x)f_Y(y) = 12xy(1-x)$, $\boxed{X \text{ and } Y \text{ are independent}}$.

b

$$
\begin{aligned}
E[X] &= \int_0^1 x f_X(x)\ dx \\
&= \int_0^1 6x^2(1-x)\ dx \\
&= \boxed{\tfrac{1}{2}}
\end{aligned}
$$

c

$$
\begin{aligned}
E[Y] &= \int_0^1 y f_Y(y)\ dy \\
&= \int_0^1 2y^2\ dy \\
&= \boxed{\tfrac{2}{3}}
\end{aligned}
$$

d

$$\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
E[X^2] &= \int_0^1 x^2 f_X(x)\ dx \\
&= \int_0^1 6x^3(1-x)\ dx \\
&= \frac{3}{10} \\
Var[X] &= \frac{3}{10} - \left(\frac{1}{2}\right)^2 \\
&= \boxed{\frac{1}{20}}
\end{aligned}$$

e

$$\begin{aligned}
Var[Y] &= E[Y^2] - E[Y]^2 \\
E[Y^2] &= \int_0^1 y^2 f_Y(y)\ dx \\
&= \int_0^1 2y^3\ dy \\
&= \frac{1}{2} \\
Var[Y] &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\
&= \boxed{\frac{1}{18} = .056}
\end{aligned}$$

2 Ch 6 Problem 26

a

$$\begin{aligned}
F(a,b,c) &= P(A \le a, B \le b, C \le c) \\
&= P(A \le a)P(B \le b)P(C \le c) \quad \text{because A, B, C are independent} \\
&= \boxed{abc}
\end{aligned}$$

b

We should first mention that the joint density of A, B, C is equal to 1. The roots of a quadratic are real when the discriminant is nonnegative, that is, when $B^2 - 4AC \ge 0$.

$$\begin{aligned}
P(B^2 - 4AC \geq 0) &= \int_0^1 \int_0^{min(\frac{1}{4c},1)} \int_{2\sqrt{AC}}^1 dB\ dA\ dC \\
&= \int_{.25}^1 \int_0^{\frac{1}{4c}} \int_{2\sqrt{AC}}^1 dB\ dA\ dC + \int_0^{.25} \int_0^1 \int_{2\sqrt{AC}}^1 dB\ dA\ dC \\
&= \int_{.25}^1 \int_0^{\frac{1}{4c}} 1 - 2\sqrt{AC}\, dA\ dC + \int_0^{.25} \int_0^1 1 - 2\sqrt{AC}\, dA\ dC \\
&= \int_{.25}^1 A - 2\sqrt{C} * \frac{2}{3} A^{\frac{3}{2}} \Big|_0^{\frac{1}{4c}} dC + \int_0^{.25} A - 2\sqrt{C} * \frac{2}{3} A^{\frac{3}{2}} \Big|_0^1 dC \\
&= \int_{.25}^1 \frac{1}{4c} - \frac{1}{6C}\ dC + \int_0^{.25} 1 - \frac{4}{3}\sqrt{C}\ dC \\
&= \left(\frac{1}{12} \ln C \right) \Big|_{.25}^1 + \left(C - \frac{8}{3} C^{\frac{3}{2}} \right) \Big|_0^{.25} \\
&= -\frac{1}{12} \ln \frac{1}{4} + .25 - \frac{8}{9} * \frac{1}{8} \\
&= \boxed{\tfrac{1}{12} \ln \tfrac{1}{4} + \tfrac{5}{36} = .2544}
\end{aligned}$$

3 Ch 6 Problem 38

a

$$\begin{aligned}
P(X = j, Y = i) &= P(Y = i | X = j) P(X = j) \\
&= \frac{1}{j} * \frac{1}{5} \\
&= \boxed{\tfrac{1}{5j} \quad 1 \leq i \leq j, 1 \leq j \leq 5}
\end{aligned}$$

b

First find the marginal mass function of Y:

$$\begin{aligned}
P(Y = i) &= \sum_{j=i}^{5} P(Y = i | X = j) P(X = j) \\
&= \boxed{\textstyle\sum_{k=i}^5 \frac{1}{k} * \frac{1}{5}}
\end{aligned}$$

$$\begin{aligned}
P(X = j | Y = i) &= \frac{P(X = j, Y = i)}{P(Y = i)} \\
&= \frac{\frac{1}{5j}}{\sum_{k=i}^5 \frac{1}{k} * \frac{1}{5}} \\
&= \frac{1}{j \sum_{k=i}^5 \frac{1}{k}}
\end{aligned}$$

In tabular form:

Table 1: Conditional distribution of X given Y

$X\backslash Y$	1	2	3	4	5
1	.440	0	0	0	0
2	.219	.390	0	0	0
3	.146	.260	.426	0	0
4	.109	.195	.319	.556	0
5	.088	.156	.255	.444	1

c

$\boxed{\text{No.}}$ For example, $P(Y = 1|X = 1) = 1 \neq P(Y = 1)$.

4 Ch 6 Problem 39

$$P(Y = j|X = i) = \frac{P(X = i, Y = j)}{P(X = i)}$$

The joint density was computed in the previous homework. The method for computing the marginal density is the same as in the previous problem. For $1 \leq i \leq 6$,

$$P(Y = j|X = i) = \begin{cases} \frac{2}{2i-1} & 1 \leq j < i \\ \frac{1}{2i-1} & j = i \end{cases}$$

X and Y are $\boxed{\text{not independent}}$. For example, $P(X = 6|Y = 6) = 1 \neq P(X = 6)$.

5 Ch 6 Problem 40

First compute the marginal mass function of Y:

$$\begin{aligned} P(Y = 1) &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \\ P(Y = 2) &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \end{aligned}$$

a

$$\begin{cases} P(X = 1|Y = 1) = \frac{p(1,1)}{P(Y=1)} = \frac{1}{8}/\frac{1}{4} = \boxed{\frac{1}{2}} \\ P(X = 2|Y = 1) = \frac{p(2,1)}{P(Y=1)} = \frac{1}{8}/\frac{1}{4} = \boxed{\frac{1}{2}} \end{cases}$$

$$\begin{cases} P(X = 1|Y = 2) = \frac{p(1,2)}{P(Y=2)} = \frac{1}{4}/\frac{3}{4} = \boxed{\frac{1}{3}} \\ P(X = 2|Y = 2) = \frac{p(2,2)}{P(Y=2)} = \frac{1}{2}/\frac{3}{4} = \boxed{\frac{2}{3}} \end{cases}$$

b

X and Y are $\boxed{\text{not independent}}$. For example, $P(X = 1|Y = 1) \neq P(X = 1|Y = 2)$.

6 Ch 6 Problem 42

We first find the marginal density of x, $f_X(x)$, and the value of the constant c in one fell swoop:

$$\begin{aligned}
1 &= \int_{-x}^{x}\int_{0}^{\infty} f(x,y)\ dy\ dx \\
&= \int_{0}^{\infty}\int_{-x}^{x} c(x^2-y^2)e^{-x}\ dy\ dx \\
&= \int_{0}^{\infty} c\left[e^{-x}\left(x^2y-\frac{y^3}{3}\right)\Big|_{-x}^{x}\right]\ dx \\
&= \int_{0}^{\infty} \underbrace{c*\frac{4}{3}e^{-x}x^3}_{f_X(x)}\ dx \quad \text{integrate by parts} \\
&= c*\frac{4}{3}*e^{-x}(x^3+3x^2+6x+6)\Big|_{0}^{\infty} \\
&= 8c
\end{aligned}$$

It follows that $c=\frac{1}{8}$ and $f_X(x)=\frac{1}{6}e^{-x}x^3$.

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{\frac{1}{8}(x^2-y^2)e^{-x}}{\frac{1}{6}e^{-x}x^3} \\
&= \frac{3}{4}\left(\frac{1}{x}-\frac{y^2}{x^3}\right)
\end{aligned}$$

7 Ch 6 Problem 44

Because X_1, X_2 and X_3 are independent, the joint density is the product of the marginals: $f(x_1,x_2,x_3)=1*1*1=1,\ 0<x_i<1,\ i=1,2,3.$

$$\begin{aligned}
P(\text{one } X_i \text{ is larger than the sum of the others}) &= 3P(X_1>X_2+X_3) \\
&= 3*\int_0^1\int_0^{1-X_3}\int_1^{X_2+X_3} 1\ dX_1\ dX_2\ dX_3 \\
&= 3*\frac{1}{6} \\
&= \boxed{\tfrac{1}{2}}
\end{aligned}$$

8 Ch 6 Problem 48

a

$$\begin{aligned} P(\min(X_1,\ldots,X_5) \le a) &= 1 - P(\min(X_1,\ldots,X_5) > a) \\ &= 1 - P(X_1 > a, \ldots, X_5 > a) \\ &= 1 - \prod_{i=1}^{5} P(X_i > a) \quad \text{by independence} \\ &= 1 - \prod_{i=1}^{5} (1 - \left(1 - e^{-\lambda a}\right)) \\ &= 1 - e^{-5\lambda a} \end{aligned}$$

b

In order for the maximum of the five random variables to be less than a[1]

$$\begin{aligned} P(\min(X_1,\ldots,X_5) \le a) &= P(X_1 \le a, \ldots, X_5 \le a) \\ &= \prod_{i=1}^{5} P(X_i \le a) \quad \text{by independence} \\ &= \prod_{i=1}^{5} \left(1 - e^{-\lambda a}\right) \\ &= \boxed{\left(1 - e^{-\lambda a}\right)^5}. \end{aligned}$$

[1]For example, let a = 50, and let X_i equal your homework score on your ith to last assignment. If you violated the rules of engagement and copied down answers from a solutions website, then $max(X_i)$ is now $\le a$ in the grades spreadsheet.