Stochastic Combinatorial Optimization from TSPs to MSTs via Caterpillars and Dogerpillars

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A Path Through the TSP, MST, and Interpolations

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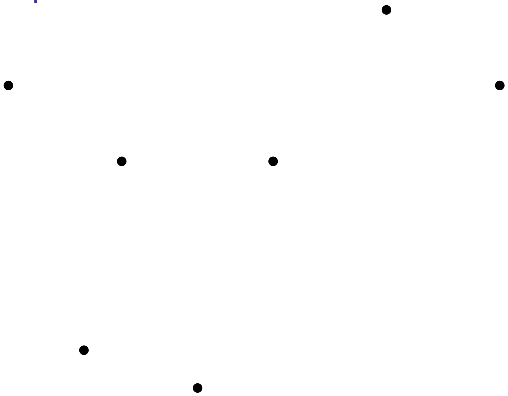
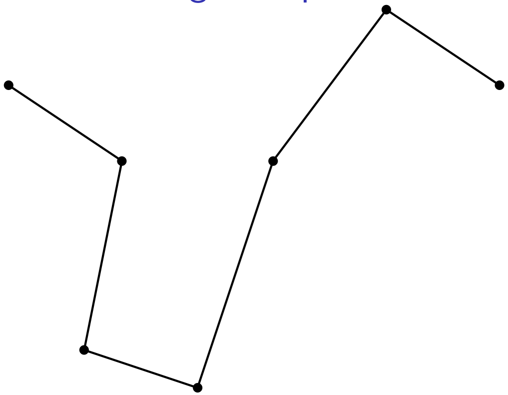
- Part III: Interpolation — The Real Theme
  - (Wherein the famous Dogerpillars are introduced and explored.)

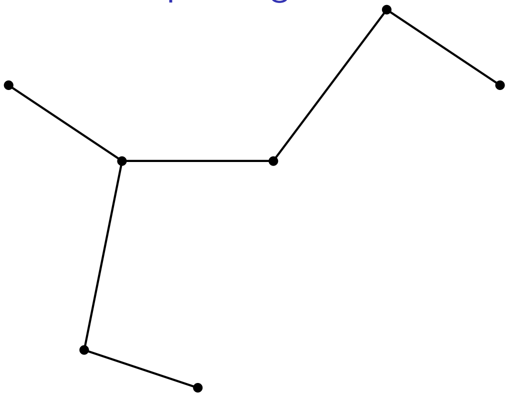
Consider a set of points ...



Now, a Shortest Path through the points ...



Now, consider a minimal spanning tree ...



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- On the Other Hand:
  There is a $O(n)$ time ϵ algorithm (Karp-Steele (1985)) if you assume a probability model for the points.

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- BUT:
  Their computational theory tells us that they are wildly different

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- For the TSP this is the famous Beardwood-Halton-Hammersly theorem of 1959. For the MST the result is from Steele (1988). The constants C_{TSP} and C_{MST} are not known exactly. The natural analogs hold in $d \geq 2$.

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- The modern package is much more robust to changes in "problem" and "model".

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- Even now this may seem surprising. Here, and in many other cases, it gives an very pleasing path to the desired strong laws.

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- We do not know if this is the "truth" when $d > 2$. Lower bounds on variance are hard to come by. We have similar open issues with respect to sharp concentration in $d > 2$.

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- Three Hints Do It: Cyclic processes, $O(n^{-(1+\epsilon)/2})$ shifts, and subsequent scales.

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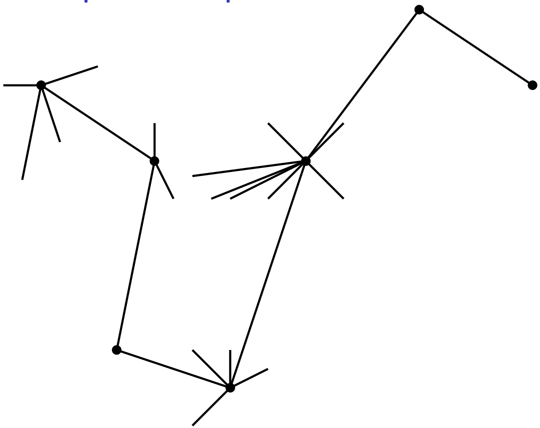
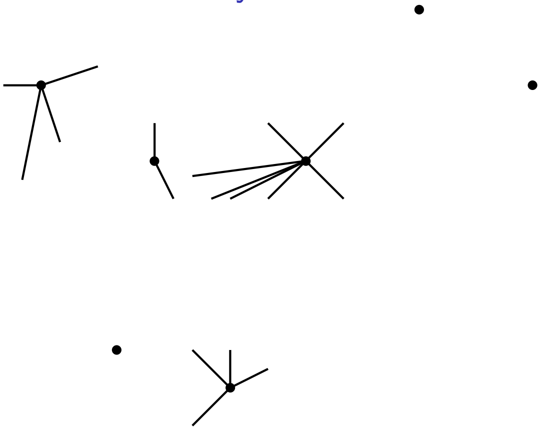
- The minimal spanning caterpillar is well defined and we naturally have

$$L_n^{\text{MST}} \leq L_n^{\text{CAT}} \leq L_n^{\text{TSP}}$$

A Picture of a Simple Caterpillar ...



Remove a path and have only stars ...



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- Still, this is good enough. One gets the strong law for minimal spanning caterpillars.

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- Ironically, if you Google "dogerpillar" you still find much that irrelevant. It's hard to come up with a neologism these days.

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- Specializations:
  - Taking $k = n$, we see that the Dogerpillar is a tree
  - Taking $k = 0$, we see that the Dogerpillar is a path
  - The most interesting cases are $k = o(n)$, especially $k = O(\sqrt{n})$.

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  - At the critical rate, we have the CLT, but we may be able to get the CLT at much lower critical rates.

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  - At the critical rate, we have the CLT, but we may be able to get the CLT at much lower critical rates.
- Let’s look at the progress to date...

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  - CLT for at least some interesting ranges of k_n, at least $k_n \sim n^{1/2+\epsilon}$

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- Thank You for Your Attention!