Martingale Markets: Abstracting the Distinguished Asset

J. Michael Steele

June 26, 2008
Appealing Theory, Appalling Facts, and Eternal Hope
Part I: Rubinstein’s Theorem and the CAPM
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- Part I: Rubinstein’s Theorem and the CAPM
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- Part III: “Natural” Models where the Market Asset is Special
- Part IV: Mathematical Features of a Candidate
- Part V: Return to the Menagerie and a Free Snack
Consider a one-period investment model. Assume each investor has a utility function and acts to maximize his next period expected utility. Assume that asset returns are jointly normal. Big CAPM Conclusion: For each asset $A$, $r_A - r_0 = \beta (r_M - r_0) + \epsilon$ where $\epsilon \sim N(0, \sigma)$.
Consider a one-period investment model

- Assume each investor
- Has a utility function
- Acts to maximize his next period expected utility
- Assume that asset returns are jointly normal

Big CAPM Conclusion: For each asset $A$

$$r_A - r_0 = \beta (r_M - r_0) + \epsilon$$

where $\epsilon \sim N(0, \sigma)$

$r_A$ is the return on the asset, $r_0$ is the risk-free rate, $r_M$ is the market return, and $\beta$ is a constant that depends on all the utilities and on the distribution of the asset returns.
Rubinstein’s CAPM Theorem

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CAPM: Why So Many So Love It

- All your investment problems reduce to one problem
- You will hold all assets in proportion to their market weight
- The "market" and the risk free asset are the only assets you need to consider: Pick your "percent" and you are done.

- This conclusion is massively appealing!
- Moreover, it is simply mathematics...
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What Do You Mean by “Distribution of Returns”?

There is a subtle assumption implicit even in speaking about “the distribution of returns.” It always makes sense to speak of distribution of \( r_t \) given the past \( r_{t-1}, r_{t-2}, \ldots \), but to speak of the distribution of \( \{r_t\} \) by itself, we must assume stationarity.

We can’t actually test for stationarity. Example: Consider any deterministic cycle with a random start.

As a matter of practice, this doesn’t matter much. As an intellectual matter, there is strangely good news.

Common Sense (of Sorts): One should only assume that which one cannot test and reject.

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Testing the Normality of Asset Returns

▶ Take any decent sized time series of almost any asset — Stock, Bond, Mutual Fund, ETF, or more exotic item.
▶ Take any test of normality: Jarque-Bera, Shapiro-Wilks, even Kolmogorov-Smirnov...
▶ You will almost always strongly reject the normality of the returns. With a test that is tail sensitive, such as Jarque-Bera, rejection is a virtual certainty.
▶ Bottom Line: Asset Returns are not normal.
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Asset Returns — The First Stylized Facts:
- Fatter Tails — more like a T with 3 to 5 degrees of freedom
- Modest Asymmetry — Left tail is fatter than the right tail
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Take the returns of a common stock and apply a test such as Ljung-Box that measures the distance from white noise. You typically fail to reject the white noise hypothesis. This modestly argues that perhaps the independence assumption of the Black-Scholes world is not so bad?

Here we come to a strange but creative idea —- On a whim, consider the squares of the returns. The tests for linear predictability (ACF tests, LB tests) now show massive predictability — hence massive dependence of the series \( \{ r_t^2 \} \). Second Stylized Fact: Asset returns are not independent. At a minimum their squares show substantial predictability.
Pondering the Independence of Asset Returns

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More Stylistic Facts

High volatility begets high volatility (ARCH effect)

Large negative shocks tend to produce a greater increase in volatility than positive shocks of comparable size. (Black's "Leverage effect")

A major portion of individual stocks' movements are explained by the movement of the overall market (CAPM effect)

Almost ninety percent of a stock's movement can be explained by the market movement and two other factors

The change in BMS, a zero cost portfolio of big cap minus small cap stocks (Small Cap Effect)

The change in HML, a zero cost portfolio of high B/M stocks minus small B/M stocks (Value Effect)

The stochastic features of asset returns may possess many mysteries, but there are also consistent behaviors found across different nations, across different asset classes, and over many different time periods and time scales.

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- The stochastic features of asset returns may possess many mysteries, but there are also consistent behaviors that are found across different nations, across different asset classes, and over many different time periods and time scales.
Sidebar on Black-Scholes World

In the Black-Scholes World we assume that the stock price evolves according to:

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \]

This implies that day \( t \) returns \( r_t = \log(S_t/S_{t-1}) \) are normally distributed and that they are independent.

We're prepared to make assumptions that have weak spots, but we typically expect our models to be approximately realistic at least at some level.

There is much interesting history and sociology in the Black-Scholes trajectory.

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Use and Non-Use of Stylized Facts

Suppose we consider a new probabilistic model....

We should feel happy when it captures stylized facts — especially critical one or subtle ones. We should face squarely those facts that are not captured by the model.

News Flash: People are not always forthright in this respect: Essentially all pension funds explicitly or implicitly assume independence of annual returns. They also assume return rates and volatilities are well estimated under the model of IID returns.

It is odd that we impute so much "efficiency" to markets where the biggest players are so tangled up in their own pajamas.
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What Is the Fundamental Question?

We love the CAPM, but we certainly can't buy Rubinstein’s assumptions, and we are unhappy with the myriad of CAPM tests. Still, we have some faith. The market asset really is special, by golly. It may be possible to extract this experience from a model that does not violate a horrible list of stylistic facts. We can hunt for this model by leaning hard on generality and abstraction.
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Martingale Markets, or MarketGales

▶ We take as given a collection $T$ of assets (stochastic processes).

▶ We assume there is a distinguished asset $V = \{V_t\} \in T$. This could be the market asset, but it need not be.

▶ Given a constant $0 \leq k < \infty$, we say the triple $(T, V, k)$ is a martingale market provided that for each $S = \{S_t\} \in T$ the process $J_t(S, V, k)$ defined by $J_t(S, V, k) = S_t - k \int_0^t 1_{V_u} \langle S, V \rangle_u \, du$, $0 \leq t \leq T$, is an $F_t$ martingale.

▶ Admittedly, this is strange, but bear with me. I'll at least show it is interesting.
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  - Uniqueness: If $(\mathcal{T}, V)$ and $(\mathcal{T}, V')$ are MarketGales, then there is a constant $c$ such that $V_t = cV'_t$ with probability one for all $0 \leq t \leq T$. 
First Interpretation of $k$

Consider the special case $dV_t = \mu_t V_t dt + V_t \sigma_t \cdot dB_t$ and calculate

$$dJ_t(V, V, k) = dV_t - k \frac{1}{V_t} d \langle V, V \rangle_t$$

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- The instantaneous drift increases as the volatility increases.
Volatility Discounted Utility: Second View of $k$

For $1 < k < \infty$, the classical isoelastic utility function

$$U_k(w) \equiv \frac{w^{1-k}}{1-k}$$  \hspace{1cm} (1)

has Arrow-Pratt relative risk aversion $-wU'_k(w)/U''_k(w) = k$.

**Definition (Volatility Discounted Utility)**

For a martingale market $(\mathcal{T}, V, k)$ and $1 < k < \infty$ the volatility discounted utility, $D(w) \equiv D_{k,V,t}(w)$, is the map from $\mathbb{R}^+$ to $\mathbb{R}$ that is defined by

$$D_{k,V,t}(w) \equiv U_k \left( w \exp\left( -\frac{k}{2} \langle \log V, \log V \rangle_t \right) \right).$$
The Martingale Properties of Volatility Discounted Utilities

Theorem (Discounted Martingale Theorem)
If the triple \((T, V, k)\) is a martingale market, then for each \(S \in T\),

the process \(\{D(S_t) : 0 \leq t \leq T\}\) is a supermartingale and

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It's maddenly difficult to show market inefficiency by exhibiting a "superior strategy" to holding the total market. Ironically, it is easy to go the other direction — many billions are invested in provably inferior strategies.

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In theory, dominated assets cannot exist.

Poster Child of Dominated Asset: An S&P500 Index fund with 2.75% expense ratio.

Many large firms exist that offer only dominated assets.

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Finally, Your Free Snack: Odd-Lot Preference Game

In most situations the non-institutional investor is at a disadvantage — most notably with short sales or leveraged transactions. In a few situations, the individual does have an advantage, e.g. there is usually no market impact cost to trades. There is one situation that is much more concretely juicy: Tender Offers with the "Odd Lot Preference." Many times per year there are tender offers for defined quantities of shares (i.e. not "all shares") Usually these are over subscribed and those who tender only get the deal on pro-rated fraction of their shares There is usually (check!) a preference given to odd lot holders This produces a "game" where the small investor has a concrete advantage over bigger players. Implementation (Google alerts, Sec.gov, no-fee-for-tenders broker, clock awareness) Hence Your Free Snack — worth perhaps $2K-$10K year, just a nibble, but still, why not have a snack?
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