Optimal Sequential Selection
Monotone, Alternating, and Unimodal Subsequences

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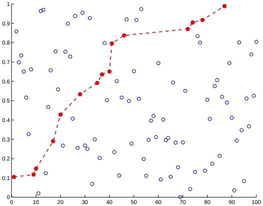
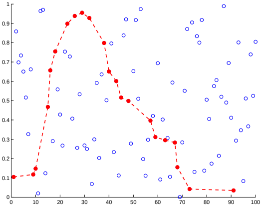
Optimal Sequential Selection of Subsequences

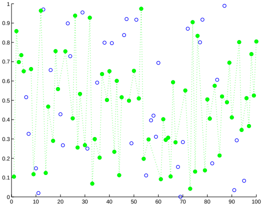
Increasing



Unimodal



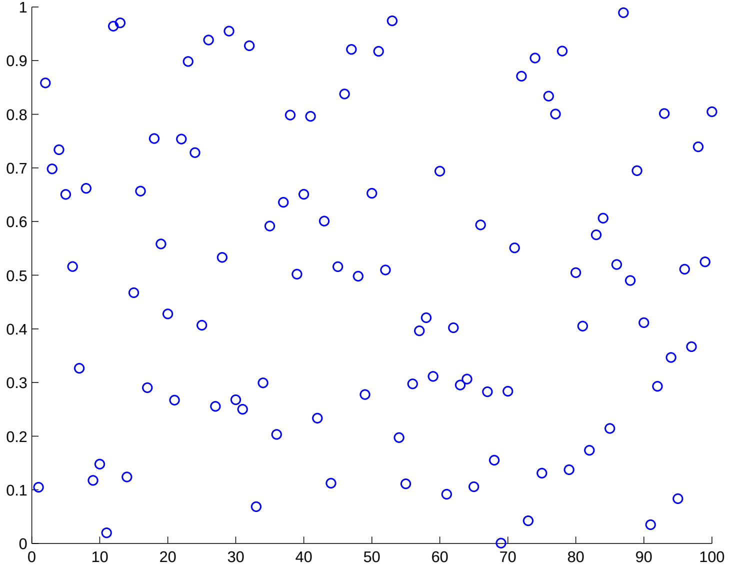
Alternating



Increasing, Unimodal and Alternating plots

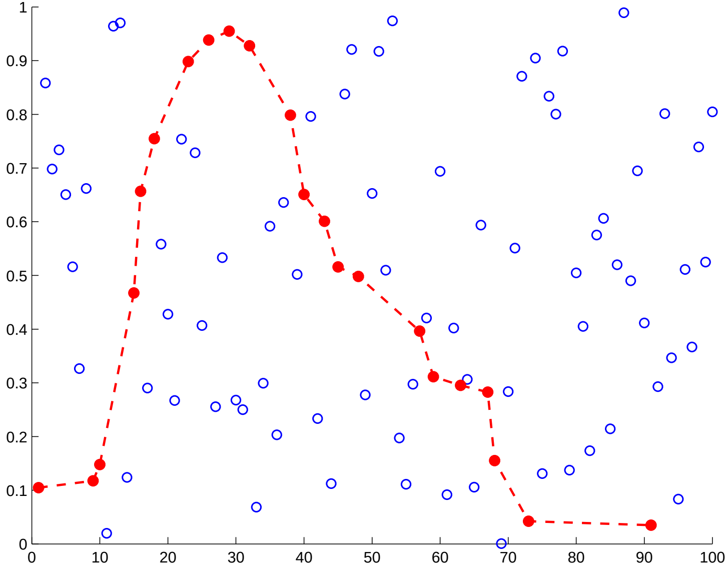
On-line vs. full-information

$n = 100$,



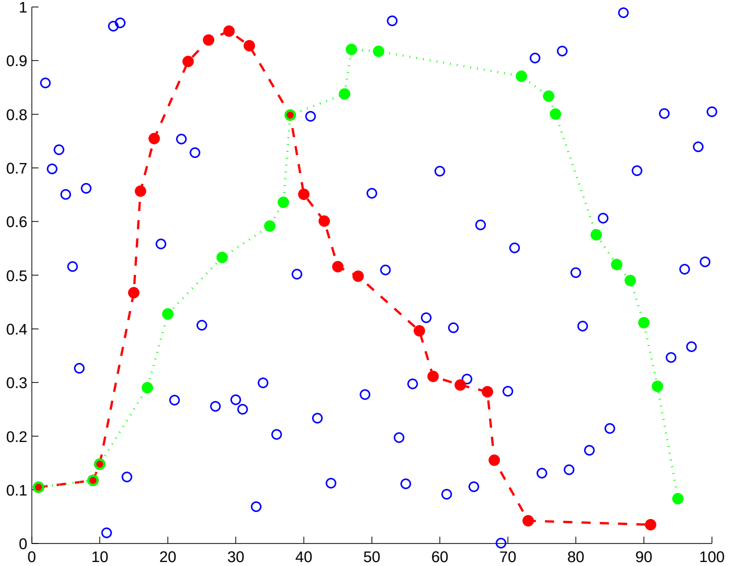
On-line vs. full-information

$n = 100, \qquad U_n^o(\pi_n^*) = 21,$



On-line vs. full-information

$n = 100,\quad U_n^o(\pi_n^*) = 21,\quad U_n = 22.$



Increasing Subsequences: Beginning with the Classics

Theorem

There is a policy $\pi^* \in \Pi(n)$ *such that* $\mathbb{E}[I_n^o(\pi^*)] = \sup_{\pi \in \Pi(n)} E[I_n^o(\pi)]$, *and for such an optimal policy one has*

$$(2n)^{1/2} - (8n)^{1/4} - 2 < \mathbb{E}[I_n^o(\pi^*)] < (2n)^{1/2} \quad \text{for all } n \geq 1.$$

so, in particular, one has

$$\mathbb{E}[I_n^o(\pi^*)] \sim (2n)^{1/2} \quad \text{as } n \to \infty.$$

- Asymptotic behavior: Samuels and Steele (1981)
- Upper bound: Bruss and Robertson (1991), Gnedin (1999)
- Lower bound: Rhee and Talagrand (1991)
- Well Trod Ground but with Something New:
  - Different proof for the upper-bound;
  - Variance bounds

Unimodal Subsequences: More Complex but Still Analogous

Theorem

There is a policy $\pi^* \in \Pi(n)$ *such that*

$$\mathbb{E}[U_n^o(\pi^*)] = \sup_{\pi \in \Pi(n)} E[U_n^o(\pi)],$$

and for such an optimal policy there is a constant C *such that*

$$2n^{1/2} - Cn^{1/4} < \mathbb{E}[U_n^o(\pi^*)] < 2n^{1/2} \quad \text{for all } n \geq 1.$$

So, in particular, one has

$$\mathbb{E}[U_n^o(\pi^*)] \sim 2n^{1/2} \quad \text{as } n \to \infty.$$

Alternating Subsequences: Something Quite Different

Theorem (Asymptotic Selection Rate for Large Samples)

For each $n = 1, 2, \ldots$, *there is a policy* $\pi_n^* \in \Pi$ *such that* $\mathbb{E}[A_n^o(\pi_n^*)] = \sup_{\pi \in \Pi} \mathbb{E}[A_n^o(\pi)]$, *and for such an optimal policy one has for all* $n \geq 1$ *that*

$$(2-\sqrt{2})n \leq \mathbb{E}[A_n^o(\pi_n^*)] \leq (2-\sqrt{2})n + C,$$

where C *is a constant with* $C < 11 - 4\sqrt{2} \sim 5.343$. *In particular, one has*

$$\mathbb{E}[A_n^o(\pi_n^*)] \sim (2-\sqrt{2})n \quad \text{as } n \to \infty.$$

Theorem (Expected Selection Size in Geometric Samples)

For each $0 < \rho < 1$, *there is a* $\pi^* \in \Pi$, *such that* $\mathbb{E}[A_N^o(\pi^*)] = \sup_{\pi \in \Pi} \mathbb{E}[A_N^o(\pi)]$, *and for such an optimal policy one has*

$$\mathbb{E}[A_N^o(\pi^*)] = \frac{3 - 2\sqrt{2} - \rho + \rho\sqrt{2}}{\rho(1-\rho)} \sim (2-\sqrt{2})(1-\rho)^{-1} \sim (2-\sqrt{2})\mathbb{E}N \quad \text{as } \rho \to 1.$$

Proof of the Expected Length of Alternating Subsequences (sketch)

- Finite-horizon Bellman equation:

$$v_{i,n}(s,r) = \begin{cases} s v_{i+1,n}(s,0) + \int_s^1 \max\{v_{i+1,n}(s,0), 1 + v_{i+1,n}(x,1)\}\,dx & \text{if } r=0 \\ (1-s) v_{i+1,n}(s,1) + \int_0^s \max\{v_{i+1,n}(s,1), 1 + v_{i+1,n}(x,0)\}\,dx & \text{if } r=1 \end{cases}$$

- Reflection identity: $v_{i,n}(s,0) = v_{i,n}(1-s,1)$ for all $1 \le i \le n$ and all $s \in [0,1]$.
- "Flipped" finite-horizon Bellman equation:

$$v_{i,n}(y) = y v_{i+1,n}(y) + \int_y^1 \max\{v_{i+1,n}(y), 1 + v_{i+1,n}(1-x)\}\,dx.$$

- "Flipped" infinite-horizon Bellman equation:

$$v(y) = \rho y v(y) + \int_y^1 \max\{\rho v(y), 1 + \rho v(1-x)\}\,dx.$$

- Threshold-policy for infinite-horizon: $f^*(y) = \max\{\xi_0, y\}$, $\xi_0 \in [0, 1/2)$
- Solve for $v(\cdot)$ and obtain

$$v(0) = v(\xi_0) = \frac{3 - 2\sqrt{2} - \rho + \rho\sqrt{2}}{\rho(1-\rho)}.$$

Proof of the Expected Length of Alternating Subsequences (sketch)

- Finite-horizon lower bound: use the infinite-horizon threshold policy.
- Finite-horizon upper bound: use the finite-horizon optimal threshold functions $\{f^*_{1,n}, \ldots, f^*_{n-2,n}\}$ and regenerate this selection process over an infinite horizon. The value of $\mathbb{E}[A^o_N(\pi^*)]$ then gives the desired upper bound.

The News You Can Use

- First — Soften The Ground:
- Study the more “symmetrical” infinite horizon (or other “smoothed”) problem variations
- Second — Have the Courage (and Techniques) to Return to Finite n:
- Typically, it is the finite n problem that interests us most. We can try to return to finite n by exploiting “suboptimality”.
- This works well enough for means but more refined information (such as variance) requires much more work.
- Open Problems:
  - CLT for “Monotone” and Finite n?
  - CLT for “Alternating” (even good variance asymptotics)
  - Richer Understanding of Martingale connections with the Bellman Equation
  - Richer Understanding of Bellman Equation asymptotics (and max-type integral equations)

Thank you!

References

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