Reflections:
Sequences, Subsequences, and Intersections

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Duke University 2015

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- Ah, hell, its “my birthday” — one small indulgence can’t hurt!

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- This theorem crushes many questions about stationary processes.

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$$n!\rho_n^n \quad \text{atoms — and one has} \quad \rho_n \to 0.$$

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  In other words, a super-exponentially small fraction will suffice. We also showed that even if the process has *zero* entropy then *one can not improve on this bound.*
- The proofs are cool (using the DeBruijn necklace at one point) and both the SMB and permutations are important. Still, opportunity calls —

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- And for dessert?
- When possible, I always enjoy "stochastic consequences" without "direct stochastic assumptions".

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- If $k = \max(a_i, b_i)$ there are at most k^2 labels — hence at most k^2 points in our sequence.

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- What about non-random sequences that share some of the properties of i.i.d uniform? Specifically, the Weyl sequences or the van der Corput sequence?

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- The behavior of the LIS depends on the continued fraction expansion of α.
- For example: $\alpha = (1 + \sqrt{5})/2 = [1; 1, 1, \ldots]$ one has

$$\limsup_{n\to\infty} L(\alpha, 2\alpha, \ldots, n\alpha)/\sqrt{n} = 5^{1/4}$$

$$\liminf_{n\to\infty} L(\alpha, 2\alpha, \ldots, n\alpha)/\sqrt{n} = 2/5^{1/4}$$

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- This determines a sequence $v_1, v_2, \ldots$: The van der Corput sequence; the most smoothly distributed sequence. About 40 years ago Andres del Junco and I found that

$$\limsup_{n\to\infty} L(v_1, v_2, \ldots, v_n)/\sqrt{n} = 3/2$$

$$\liminf_{n\to\infty} L(v_1, v_2, \ldots, v_n)/\sqrt{n} = \sqrt{2}$$

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  - The Erdös Szekeres Theorem is published.

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- Alessandro Arlotto, Vinh Nguyen, and I found most recently that $L^{SEQ}(X_1, X_2, \ldots, X_n)$ satisfies the natural CLT; this is in total contrast to $L(X_1, X_2, \ldots, X_n)$ — the Global Optimum.

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- Despite a huge literature, there is relatively little that corresponds to probability "Laws" — though algorithms and models abound.
- This yields a rich soup of problems and examples. Here one is reminded of an observation of Conway....

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  - Special problems are *not so special*; every special problem is a test of technique or an invitation to the development of deeper techniques.

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- This has been a treat — Thanks!