Optimal Sequential Selection
Alternating Subsequences: Means, Concentration, and CLTs

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Introduction and Motivation

- Famous combinatorial problems with long mathematical history on sequences of n real numbers, or permutations of the integers $1, \ldots, n$
  - Erdős and Szekeres (1935): monotone subsequences
  - Fan Chung (1980): unimodal subsequences
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  - Longest monotone subsequences: Hammersley (1972), Kingman (1973), Logan and Shepp (1977), Veršik and Kerov (1977), . . .
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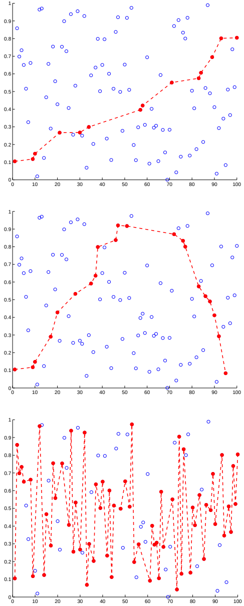

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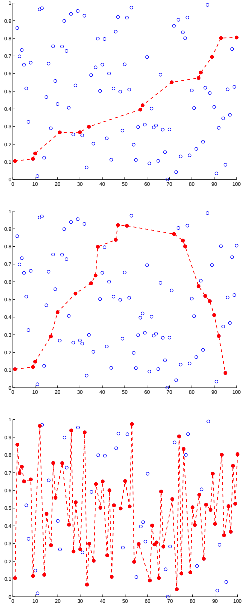
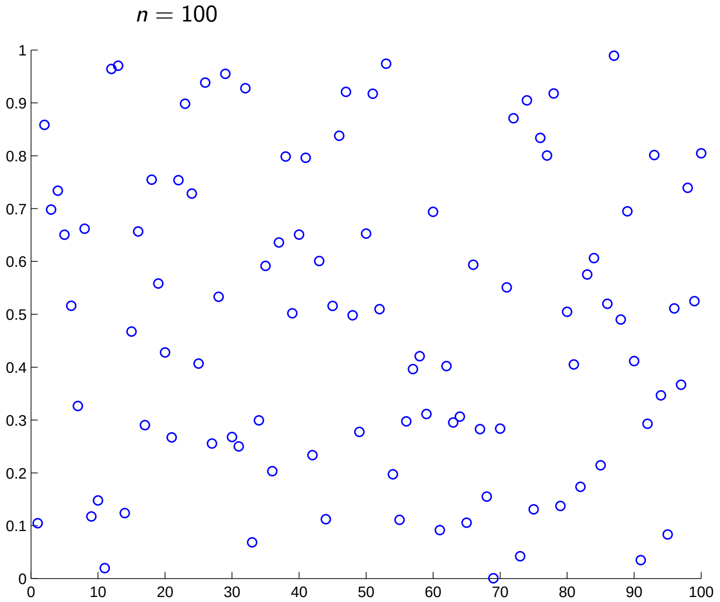
- Study the sequential (on-line) version of these problems
  - Objective: maximize the expected length (number of selections) of monotone, unimodal and alternating subsequences



Full-information vs. on-line — Increasing



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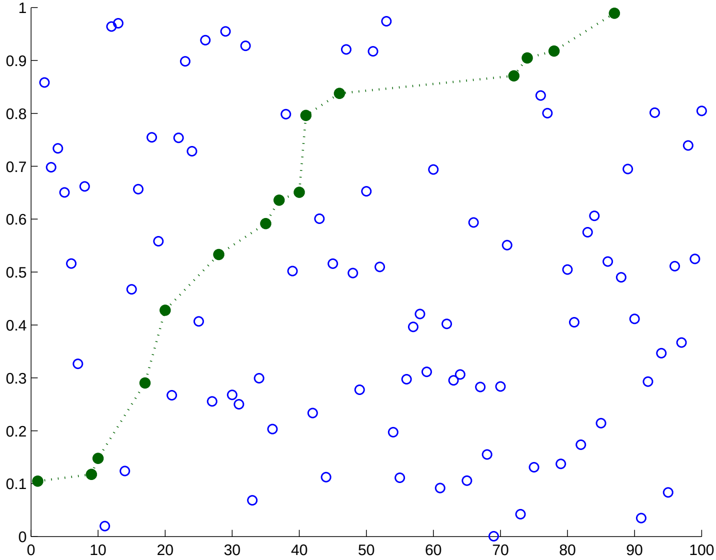
$n = 100 \qquad l_n = 15$



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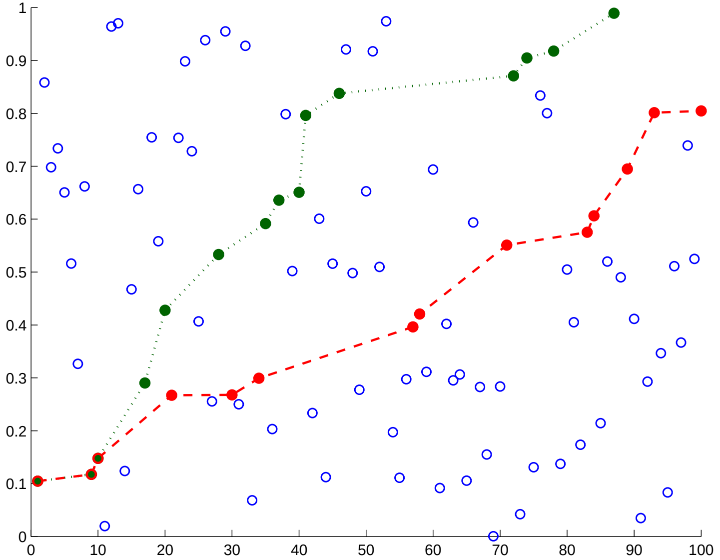
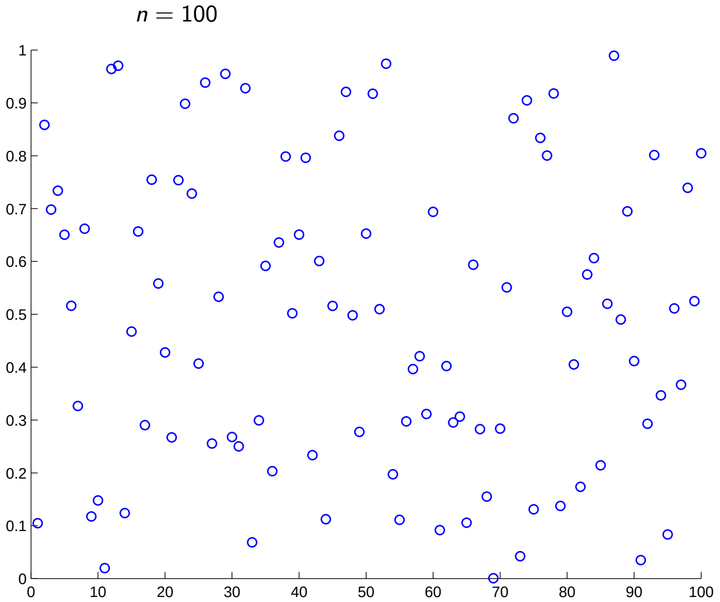
$n = 100$ $\quad$ $l_n = 15$ $\quad$ $l_n^o(\pi_n^*) = 14$



Full-information vs. on-line — Unimodal



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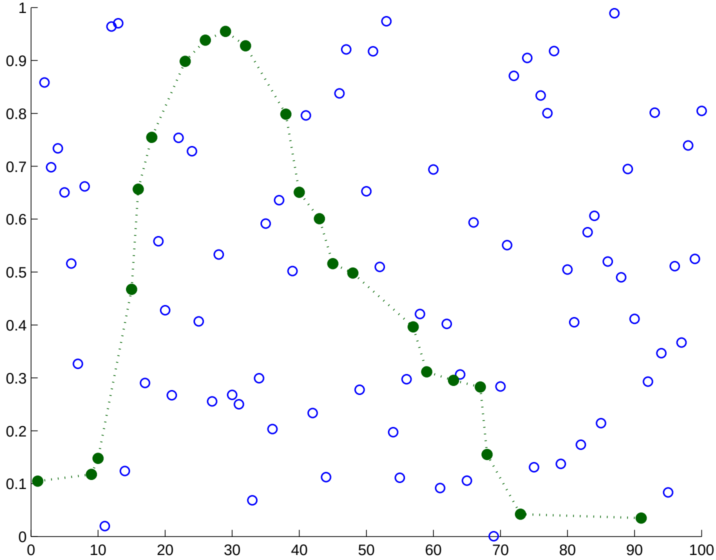
$n = 100$ $U_n = 22$



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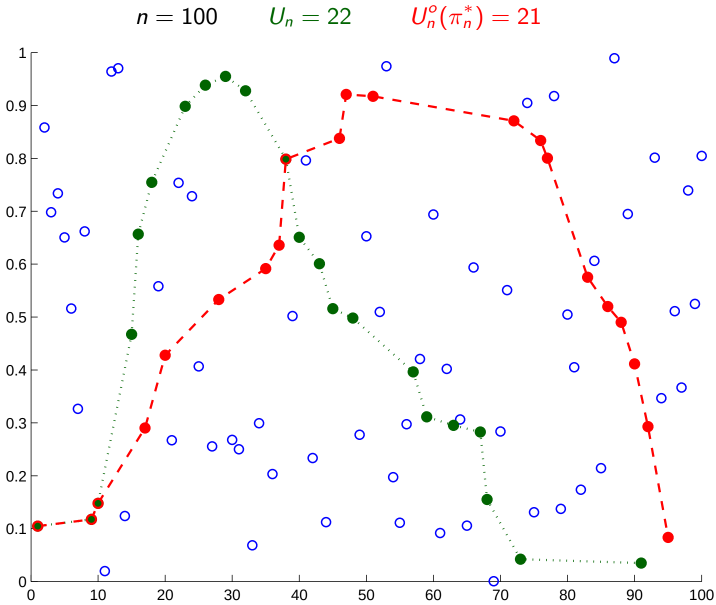

Increasing Subsequences: Beginning with the Classics

Theorem (On-Line Monotone: The Leading Case)

There is a policy $\pi^* \in \Pi(n)$ *such that* $\mathbb{E}[I_n^o(\pi^*)] = \sup_{\pi \in \Pi(n)} E[I_n^o(\pi)]$, *and for such an optimal policy and all* $n \geq 1$ *one has*

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Theorem (Something New – Variance Bounds (Arlotto & Steele, 2011))

For all $n \geq 1$, *one has*

$$\mathbb{E}[I_n^o(\pi^*)]/3 - 2 \leq \mathrm{Var}[I_n^o(\pi^*)] \leq \mathbb{E}[I_n^o(\pi^*)].$$

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- Bin-packing Connection: SMS is cognate to a special bin packing problem, and the proof of this variance bound applies to a *rich class* of these.

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Theorem (Arlotto & Steele, 2011)

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- MDP Connections: Here we have a second MDP where "the mean bounds the variance." This and further examples promise the beginning of a handy theory that knits all the examples together.

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where C *is a constant with* $C < 11 - 4\sqrt{2} \sim 5.343$.

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Sketch of the Tools and Methods: Alternating Subsequence Problem

- Finite-horizon Bellman equation:

$$v_{i,n}(s,r) = \begin{cases} s v_{i+1,n}(s,0) + \int_s^1 \max\{v_{i+1,n}(s,0), 1 + v_{i+1,n}(x,1)\}\, dx & \text{if } r = 0 \\ \end{cases}$$

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$$v_{i,n}(y) = y v_{i+1,n}(y) + \int_y^1 \max\left\{ v_{i+1,n}(y), \qquad 1 + v_{i+1,n}(1-x) \qquad \right\} dx.$$

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- "Flipped" infinite-horizon Bellman equation — the "Easy One":

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- Threshold-policy for infinite-horizon: $f^*(y) = \max\{\xi_0, y\}$, $\xi_0 \in [0, 1/2)$

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$$v_{i,n}(s,r) = \begin{cases} s v_{i+1,n}(s,0) + \int_s^1 \max\left\{v_{i+1,n}(s,0), 1 + v_{i+1,n}(x,1)\right\} dx & \text{if } r = 0 \\ (1-s) v_{i+1,n}(s,1) + \int_0^s \max\left\{v_{i+1,n}(s,1), 1 + v_{i+1,n}(x,0)\right\} dx & \text{if } r = 1 \end{cases}$$

- Reflection identity: $v_{i,n}(s,0) = v_{i,n}(1-s,1)$ for all $1 \le i \le n$ and all $s \in [0,1]$.
- "Flipped" finite-horizon Bellman equation:

$$v_{i,n}(y) = y v_{i+1,n}(y) + \int_y^1 \max\left\{ v_{i+1,n}(y), \qquad 1 + v_{i+1,n}(1-x) \qquad \right\} dx.$$

- "Flipped" infinite-horizon Bellman equation — the "Easy One":

$$v(y) = \rho y v(y) + \int_y^1 \max\left\{\rho v(y), 1 + \rho v(1-x)\right\} dx.$$

- Threshold-policy for infinite-horizon: $f^*(y) = \max\{\xi_0, y\}$, $\xi_0 \in [0, 1/2)$
- Solve for $v(\cdot)$ and obtain

$$v(0) = v(\xi_0) = \frac{3 - 2\sqrt{2} - \rho + \rho\sqrt{2}}{\rho(1-\rho)}.$$

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- Finite-horizon *lower bound*: use the infinite-horizon threshold policy.
- Finite-horizon *upper bound*: use the finite-horizon optimal threshold functions $\{f^*_{1,n}, \ldots, f^*_{n-2,n}\}$ and regenerate this selection process over an infinite horizon. The value of $\mathbb{E}[A^o_N(\pi^*)]$ then gives the desired upper bound.

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- How Much Better Does a "Prophet" Do?

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- Here Alessandro Arlotto and I are happy to have some progress to report.

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Theorem (Arlotto & Steele, 2012)

There is a constant $\sigma > 0$ *such that*

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- Conditions to Check? These are surprisingly concrete, even though a kind of alpha mixing is involved.
- Source of Juice? We have honestly independent blocks (of random size) and this gives us all the mixing we need.

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- Thank You for Your Attention!

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