How a False Probability Model Changed the World: Birth, Death, and Redemption of Black-Scholes

J. Michael Steele

March 12, 2008
Introduction: The Special, The Empirical, The Miracle

Part I: What is (Almost) Unique to Financial Modeling — The Notions of Arbitrage and Replication

(Homework: Perhaps these notions are not so unique to financial modeling. If not, there is a long way to run.)

Part II: The Theme of "Stylistic Facts" — Something that Should be Universal in More Ways than One

(More Homework: The assignment should be easier this time.)

Part III: When Models Shape Markets

(Enough Homework: This is more of a celebration, reflection, and — perhaps — a caution.)
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Part I: Beginning with an Almost Impossible Question

Consider a world where there is a stock and a “contingent claim”.

The stock costs 2 dollars at time zero, and at time 1 it is worth

- either 4 dollars (if it goes up)
- or 1 dollar (if it goes down)

The claim is costs $X$ dollars at time zero, and at time 1 it is worth

- either 3 dollars (if the stock goes up)
- or 0 dollars (if the stock goes down)

Question: What is $X$?
Sure! Let $P_{UP}$ denote the probability that the stock goes up. In that case, a pretty reasonable price for the contingent claim would be

$$X_{\text{guess}} = 3 \cdot P_{UP} + 0 \cdot (1 - P_{UP}) = 3 \cdot P_{UP}$$
Some Reasoning about the Almost Impossible Question

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On second though, a little bit better guess would be

$$X_{better} = P_{UP} U(W + 3) + (1 - P_{UP}) U(W)$$

where $U$ is my personal utility and $W$ is my personal wealth.

Bad News: Nobody knows $P_{UP}$. It looks like we are stuck, and we all should soak for a moment in a bath of hopeless despair!

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▶ Maybe we can “replicate the contingent claim” with a “portfolio” consisting of $\alpha$ units of the stock $S$ and $\beta$ units of the bond $B$. 

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- This turns out to be a marvelously fecund idea.
Solving For X

Table: Replication of a Derivative Security

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What a nice set of equations! In our heads, we can solve to find $\alpha = 1$ and $\beta = -1$.

Corollary: $X = \alpha S + \beta B = 1 \ast 2 + (-1) \ast 1 = 1$.

Bottom Line: The unique arbitrage-free price for the contingent claim $X$ is one dollar.
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What Have We Won; What Did We Pay?

We found the "only feasible price" for $X$ and we needed no probability to do so!

Since we squeezed out the probability theory, we also squeezed out the utility theory. This is a huge win.

Otherwise different agents would offer different prices and a whole bird's nest of economic modeling would be needed to squeeze out one final market price.

The theory is enforceable. We can win money risk-free from anyone who trades at any price other than the one we derived.

SUPER BONUS. This extremely simple example carries through to the real world.
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A Streetwise Gambler Makes a Guess

We used replication and arbitrage to get a value for the contingent claim, but there is a way a streetwise gambler could have guessed the answer. If we know $P_{UP}$, then we have a good rough and tumble "guess" for the value of the contingent claim — take the expected value of the contingent payouts. The streetwise gambler has a way to "infer" a probability $P'_{UP}$ that the stock goes up. The gambler "assumes" that the stock price is a martingale: This gives $2 = P'_{UP} \times 4 + (1 - P'_{UP}) \times 1$ so $P'_{UP} = \frac{1}{3}$. The gambler then guesses $X = P'_{UP} \times 3 + (1 - P'_{UP}) \times 0 = 1$ and his guess is RIGHT!
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An Honest Theorem — Not to be Misinterpreted

The gambler has introduced what we now call “the equivalent martingale measure.”

A THEOREM now asserts that if a unique equivalent martingale measure $P'$ exists, then the arbitrage-free price of any contingent claim is just the expected value of the claim’s payouts with respect to $P'$.

Important Nuances

“Equivalent” means — puts all the probability on the same events that got probability under the original measure. There is more modeling in this bland assumption than one might guess.

The real stock price does not (by model or by observation) “follow” the law of the equivalent martingale measure.

This recipe is often called the “risk neutral approach to option pricing”, but there is no utility theory here — and the name is major misnomer.

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Introducing Black–Scholes World

The theory of option pricing owes a fundamental debt to Fisher Black and Myron Scholes who in 1973 considered the model $P$ that now many people call “Black–Scholes World”:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \quad \text{and} \quad d\beta_t = r \beta_t \, dt$$

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- Our world has a finite horizon, $T$. Thus, $\tau = T - t$ is the “time left”.
The Streetwise Gambler and Black–Scholes World

With non-zero interest rates, the street smart gambler expects $M_t = S_t / \beta_t$ to be a martingale.

Under his "equivalent martingale measure" $P'$, we have the stock/bond equations:

$$dS_t = rS_t \, dt + \sigma S_t \, dB_t$$

and

$$d\beta_t = r \beta_t \, dt$$

The new equations depend on $r$ but not on $\mu$.

The amazing consequence is that the arbitrage-free value of ANY contingent claim will NOT DEPEND ON $\mu$.

This almost defies credibility — yet still holds water.

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The Famous Formula as a Special Case

If $H$ is any function of the stock-price path $S_t$, $0 \leq t \leq T$, we can consider a contingent claim that pays us that function of the path. For example, the maximum, or the last value, or $\text{max}(S_T - K, 0)$ in the case of the European call option.

The arbitrage free price for this claim is just $\mathbb{E}_P(S_{[0: T]})$ (1).

For the European call option, the payout function just depends on the value of the stock at the terminal time.

Given the stock price at time $t$, the conditional distribution of $S_T$ given $S_t = S$ is just a log normal, so we can easily work out the expectation (1).

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▶ For the European call option, the payout function just depends on the value of the stock at the terminal time.

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\[ S \varPhi \left( \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) - Ke^{-r\tau} \varPhi \left( \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) \]

This is just the (interest rate adjusted) value of \( E_{P'}(S_{[0:T]}) \) in its concrete form; the famous Black-Scholes formula for a European call option.
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▶ Note \( \mu \) does not appear.
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We’ve used a beautiful idea arbitrage and some beautiful tools — stochastic calculus.

Details have been omitted — but no crucial ideas
There are now many good places to learn stochastic calculus and its applications to mathematical finance, but ....

There is one we most warmly recommend:

- Friendly and honest
- Rigor without tedium
- Fun for the whole family

Sure, you could get other books, but don’t you deserve the best?
In 1975 market for equity options and other derivatives were a tiny "boutique" activity.

By 2004 the notional value of derivatives contracts exceeded $273 \times 10^{12}$ USD.

What drove this explosive development? The existence of an explicit formula? I used to think so.

More likely, the key driver was the explicit recipe for hedging. This is honest and operational, even absent a "formula."

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Volatility — and Implied Volatility

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The parameter $\sigma$ in the Black-Scholes formula is called the “volatility.” This is also the parameter the model

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \quad \text{and} \quad d\beta_t = r \beta_t \, dt$$
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Trick: Replicate the Option using a Portfolio of fractions of the stocks and the “bond.”
Invoke the “Law of one Price” and solve the equations to get the arbitrage free option price.
Apply this in continuous time: Get the Black-Scholes formula.
Note (reluctantly) that empirical volatility and implied volatility are imperfectly related.
Worry a little ... while a 273 trillion dollar market evolves in less than 30 years.
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In the Black-Scholes World we assume that the stock price evolves according to:

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \]

This implies that day \( t \) returns \( r_t = \log(S_t / S_{t-1}) \) are normally distributed and that they are independent. This tweaks our empirical curiosity. Even though we're prepared to make assumptions that have weak spots, we typically expect our models to be approximately realistic at least at some level. What is the empirical story for asset returns?
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What is the empirical story for asset returns?
There is a subtle assumption implicit even in speaking about \textit{“the distribution of returns.”} It always makes sense to speak of distribution of \( r_t \) given the past \( r_{t-1}, r_{t-2}, \ldots \), but to speak of the distribution of \( \{ r_t \} \) by itself, we must assume stationarity.

We can’t actually test for stationarity. Example 1: Randomized repetition. Example 2: Cycle with a random start.

As a matter of practice, this doesn’t matter much. As an intellectual matter, there is strangely good news.

Common Sense (of Sorts): One should only assume what one cannot test and reject.

J. Michael Steele

\textit{How a False Probability Model Changed the World: Birth, Death, and Redemption of Black-Scholes}
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Testing the Normality of Asset Returns

▶ Take any decent sized time series of almost any asset — Stock, Bond, Mutual Fund, ETF, or more exotic item.

▶ Take any test of normality: Jarque-Bera, Shapiro-Wilks, even Kolmogorov-Smirnov...

▶ You will almost always strongly reject the normality of the returns. With a test that is tail sensitive, such as Jarque-Bera, rejection is a virtual certainty.

▶ Bottom Line: Asset Returns are not normal.

Asset Returns — The First Stylized Facts:

▶ Fatter Tails — more like a T with 3 to 5 degrees of freedom

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Pondering the Independence of Asset Returns

Take the returns of a common stock and apply a test such as Ljung-Box that measures the distance from white noise. You typically fail to reject the white noise hypothesis. The modestly argues that perhaps the independence assumption of Black-Scholes world is not so bad? Here we come to a strange but creative idea —- On a whim, consider the squares of the returns. The tests for linear predictability (ACF tests, LB tests) now show massive predictability — hence massive dependence of the series $\{r_t^2\}$. Second Stylized Fact: Asset returns are not independent. At a minimum their squares show substantial predictability.
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More Stylized Facts

- High volatility begets high volatility (ARCH effect)
- Large negative shocks tend to produce a greater increase in volatility than positive shocks of comparable size. (Black’s “Leverage effect”)
- A major portion of individual stocks’ movements are explained by the movement of the overall market (CAPM effect)
- Almost ninety percent of a stock’s movement can be explained by the market movement and two other factors
- The change in BMS, a zero cost portfolio of big cap minus small cap stocks (Small Cap Effect)
- The change in HML, a zero cost portfolio of high B/M stocks minus small B/M stocks (Value Effect)
- The stochastic features of asset returns may possess many mysteries, but there are also consistent behaviors found across different nations, across different asset classes, and over many different time periods and time scales.
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Suppose we consider a new probabilistic model. We should feel happy when it captures stylized facts—especially critical or subtle ones. We should face squarely those facts that are not captured by the model. News Flash: People are not always forthright in this respect: Essentially all pension funds explicitly or implicitly assume independence of annual returns. They also assume return rates and volatilities are well estimated under the model of IID returns. The Black-Scholes Model is brutally at odds with the most fundamental stylized facts for stock returns. What’s up with that?
Normative and Behavioral Use of Stylized Facts

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Can you find “arbitrage” in your favorite problem domain:

In a competitive algorithm for a network queueing protocol on
in on-line data compression algorithms can you find a analog
of a replicating portfolio? Hard but interesting.

Take any context where there expected value is a feature of
merit. It is probably the case that
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Can an arbitrage
argument get you out of the trap?

Should you systematize the “Stylized Facts” of your favorite
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What are the stylized facts of network traffic, etc.?

How do your favorite models match up with your favorite
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Everyone does this to some extent, but there is probably a
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Part III: When Models Shape Markets

Sociology of a Mathematical Innovation


Thesis: The Black-Scholes framework changed the financial markets it aimed to model.

Not an uncommon phenomenon with economic or social theories; unique for such a mathematical theory.

Thirty Years of Experience

1973-1980 Black-Scholes fits poorly (option markets are shallow and transaction costs are high)

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post October 1987 Black-Scholes is "broken" as a direct guide to market value (but markets continue to grow as new comfort levels of risk allocation are reached)

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The challenge now: Modeling Volatility!

Cross-Fertilizing Elements:

1. Arbitrage — The "Special Light" of Mathematical Finance
2. Stylized Facts — A Universal Rough Guide

These two themes seem almost bullet proof. They should serve us very well for years to come.

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