Statistics 930: Probability Theory
Homework No. 1

Instructions. You should review some concepts from real analysis. In particular, you should review

- High dimensions and "intuition"
- Continuous derivatives
- Metric Spaces, Cauchy sequences, and Completeness of a metric space.
- Uniform continuity
- Sequential compactness (existence of convergent subsequences)
- Limsup and Liminf

You should also begin (even complete) reading Chapter 1 of the text. We will deal with this material rather quickly next week.

Problem 1. Consider four boxes with side one place adjacent to the origin where one box is in the first quadrant, one is in the second quadrant, etc. Here all four boxes are contained in the big box $[-1,1]^2$. Let each box have a disk of maximum radius inscribe in the box. There is a little space containing the point $(0,0)$ that is out side of the four disks. Draw the biggest disk that you can draw around this point such that this new disk does not intersect the old disks; call this the center disk. Now consider this problem in dimension 3, 4, ..., etc. Show that for dimension 10 and larger, there are points in the center disk that are *outside of the box* $[-1,1]^d$.

Problem 2. Suppose that $f : [0,2] \to \mathbb{R}$ is continuous. Suppose that f' exists for all $x \in [0,2]$ except possibly for $x = 1$. Suppose that there is a real number A such that $f'(x)$ approaches A as x approaches 1. *Prove or disprove* that f is differentiable at $x = 1$ and that the derivative at that point is equal to A.

Problem 3. Consider the set S_0 of all continuous functions $f : [0,1] \to [0,1]$. Define a metric on the set S_0 by setting

$$\rho(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

(a) Give an example of a sequence in S_0 that does not contain a Cauchy subsequence with respect to the metric ρ.

(b) For a given $\epsilon > 0$ consider $S_\epsilon \subset S_0$ consisting of the set of functions $f \in S_0$ such that

$$|f(x) - f(y)| \le |x - y|^\epsilon \quad \text{for all } x, y \in [0,1].$$

Show from first principles that *any* sequence in S_ϵ contains a subsequence that is Cauchy with respect to the metric ρ.

Problem 4. Suppose that a sequence $\{a_n : n = 1, 2, ...\}$ real numbers is such that $a_n \ge 1$ for all $n \ge 1$ and

$$a_{n+m} \le a_n a_m \quad \text{for all } n \ge 1, m \ge 1.$$

Show that $a_n^{1/n}$ converges as $n \to \infty$.

PROBLEM 5. Suppose that $X > 0$ and $Y > 0$ are random variables such that $E(X/Y) \leq 1$. Prove that

$$E(\sqrt{X}) \leq \sqrt{EY}$$

Note: Here X/Y is X divided by Y.

PROBLEM 6. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of positive real numbers and suppose that $\{y_n\}$ is strictly increasing and unbounded. Show that one has

$$\liminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \leq \liminf_{n\to\infty} \frac{x_n}{y_n} \leq \limsup_{n\to\infty} \frac{x_n}{y_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}.$$

COMMENT: This is a discrete version of Rolle's theorem or L'Hopital's rule — take your pick. It is most interesting when the two outside limits are equal. For the proof you might want to start by letting

$$L = \liminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}.$$

You can then give yourself an $\epsilon > 0$ and start to work on the first inequality. The second inequality is trivial and you can adapt your argument of the first inequality to prove the last one.

PHILOSOPHY. Much of probability theory deals with proofs of convergence. It is a good idea to take some time early on firm up your understanding of convergence in the simplest informative settings, especially for the complete metric spaces $\mathbb{R}$ and $C[0,1]$. Also, in probability theory one makes constant use of "subsequence arguments," so these should be mastered as quickly as possible. Finally, one of the "standard tricks" of probability theory is to prove that some sequence has a limit by showing that the limsup is equal to the liminf. This is more of a strategy than a "trick." The idea eventually becomes daily companion.

The next assignment will start to deal with the notions of probability theory, so these exercises are just for warm-up. Still, it is important that you take them seriously. If they seem too easy, take this opportunity to improve your ability to write mathematical proof that others can understand; this is a skill in itself.