Statistics 530: Probability Theory
Homework No. 2

INSTRUCTIONS. You should complete your reading Chapter 1 of Durrett. Reading is part of your homework. *It is required.* You should mainly focus on the MCT, Fatou, and DCT. You need to master these tools — their proofs, their applications. This is the main message. What I have not yet discussed in class (e.g. Fubini's Theorem and Product Measures) you might read but not be concerned about the details at this point. You should begin your reading of Chapter 2. This is an important chapter and we will live with it for a couple of weeks.

PROBLEM 1. On the probability space $(\Omega, \mathcal{F}, P)$, you have two sequences of random variables $\{X_n\}$ and $\{Y_n\}$. You are given that X_n converges to X *with probability one* and Y_n converges to Y *in probability.* Give a proof or give a counter example for each of the assertions:

- $X_n Y_n$ converges in probability to XY
- $X_n Y_n$ converges in with probability one to XY

Note: After you have understood the problem and have a good idea how to write it up, think a little more about how to write it up in a way that is as nice, clear, and uncluttered as you can muster.

PROBLEM 2. You have a sequence of (possibly dependent) random variables $\{X_n\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, and you are given that

$$EX_n^2 \leq 1 \quad \text{for all } n = 1, 2, \ldots$$

First, give a counterexample that shows $X_n/\sqrt{n}$ need not converge to zero with probability one. Second, show that if the X_n all have the same distribution (though still possibly dependent) then $X_n/\sqrt{n}$ does converge to zero with probability one. Hint: For the second part you will need the First Borel Cantelli Lemma.

PROBLEM 3. You have a sequence of random variables $\{X_n\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and we consider the following (very useful!) function

$$f(x) = \frac{e^x}{1 + e^x} \quad \text{for all } x \in \mathbb{R}.$$

Prove, or disprove by counterexample, each of the following assertions:

- If X_n converges to X in probability, then $f(X_n)$ converges to $f(X)$ in probability.
- If X_n converges to X in with probability one, then $f(X_n)$ converges to $f(X)$ with probability one.
- If X_n does not converge to X in probability, then $f(X_n)$ does not converge to $f(X)$ in probability.
- If X_n does not converge to X in with probability one, then $f(X_n)$ does not converge to $f(X)$ with probability one.

PROBLEM 4. Suppose that X_i, $i = 1, 2, \ldots$ are independent and uniformly distributed on $[0, 1]$. Let T be the first value of k, such that $X_k > X_1$. Find ET. *Note*: Even this little problem has a nice economics story. Think of the X_k as salary offers; T is then the first time you get an offer that beats the very first offer that you got. *Hint*: For this calculation you'll have to recall (and use) what you know about elementary probability. You may have to review the facts about the geometric distribution.

PROBLEM 5. We suppose the events A_1, A_2,..., A_n all have probability α, but otherwise we make no assumptions. If $n\alpha > 1$, then some pair of the events have an intersection with positive probability. This obvious fact is a probabilistic version of the famous pigeonhole principle.

Prove a more quantitative version by showing that there exist a pair of integers $1 \le i < j \le n$ such that

$$\alpha^2 - \frac{1}{n-1} \le P(A_i \cap A_j).$$

Big Hint: First note that it would suffice to show that

$$n(n-1)\alpha^2 - n\alpha \le \sum_{i,j:i\neq j} P(A_i \cap A_j)$$

and then think about proving this by applying the Cauchy-Schwarz inequality (or Jensen's inequality) to *something*.

By the way, this kind of argument is often used in combinatorics to improve (or refine) a more naive pigeonhole argument. To use this specific fact, one would need to invent some probability space where one deduces something combinatorially useful from knowing that there is a pair with $P(A_i \cap A_j)$ almost as big as α^2.

PHILOSOPHY. The objects of probability theory have many ways in which they can converge. We must simultaneously retain our intuition about what convergence "should give us" and keep a sharp eye on the what the mode of convergence we are using actually gives us. The distinction can be as large as winning or losing a large fortune. If $Z_n \ge 0$ is your fortune at time n, it is easy to have $E[Z_n] \to \infty$ while Z_n converges almost surely to zero.