**Statistics 930: Probability Theory — Homework No. 3**

INSTRUCTIONS. You should read Chapter 2 of Durrett. As usual, pay most attention to the topics we have emphasized in class. Some of this chapter has so-called "foundational material." Ironically, the relevance of this foundation can only be appreciated after one has seen some problems with real meat. You may also need to review elementary material about distributions (Cauchy, Gaussian, etc.). You will be expected to know facts from real analysis about Cauchy sequences, etc.

PROBLEM 1. Suppose that $X_1, X_2, ...$ is a sequence of independent random variables with the standard Cauchy distribution. Prove or disprove the assertions:

$$\limsup_{n\to\infty} X_n/n = \infty \quad \text{with probability one,}$$

and for $\alpha > 1$

$$\limsup_{n\to\infty} X_n/n^\alpha = 0 \quad \text{with probability one.}$$

PROBLEM 2.

- Let $f : \mathbb{R} \to \mathbb{R}$ be *any* function. Let $\mathcal{F}$ be any sigma field of subsets of $\mathbb{R}$. Let $\mathcal{F}'$ be the class of all sets of the form $B = f^{-1}(A)$ where $A$ is in $\mathcal{F}$. Show that $\mathcal{F}'$ is a sigma field.
- Let $X_n$ be any sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. Show that $Y = \sup X_n$ is a random variable; i.e. $\{\omega : Y(\omega) \in B\} \in \mathcal{F}$ for each Borel set $B$.

PROBLEM 3. Suppose that $T_n$, $n = 1, 2, ...$ is a sequence of *dependent* random variables for which there is a constant $c$ such that for all $1 \le n \le m$ and all $t > 0$ one has the bound

$$P(|T_m - T_n| \ge t) \le c/(t^2 n). \tag{1}$$

Show there is a subsequence $n_k$ such that $n_{k+1}/n_k \to 1$ and such that with probability one the sequence $\{T_{n_k}\}$ is a Cauchy sequence.

PROBLEM 4. Suppose that $X_n$, $n = 1, 2, ...$ is a sequence of independent random variables such that $P(X_n = 1) = P(X_n = -1) = 1/2$. Consider the random variables defined by

$$T_n = \sum_{k=1}^{n} \frac{X_k}{k}$$

and prove there is a constant $c$ such that the inequality (1) holds.

PROBLEM 5. Suppose that $X_n$, $n = 1, 2, ...$ is a sequence of independent random variables such that $P(X_n \ge t) = e^{-t}$ for all $t \ge 0$. Let

$$Z = \limsup_{n\to\infty} \frac{X_n}{\log n}.$$

Guess the distribution of $Z$ and prove that your guess is correct. Hint: For a complete proof you will very likely need *both* of the Borel-Cantelli lemmas.

PROBLEM 6. Suppose that $X_n$, $n = 1, 2, ...$ is a sequence of independent random variables such that $P(X_n = 0) = P(X_n = 1) = 1/2$. The sum

$$Z = \sum_{k=1}^{\infty} 2^{-k} X_k$$

converges for all $\omega$, not just with probability one. Guess the distribution of this random variable and prove that your guess is correct.

Problem 7. Suppose that $X_n$, $n = 1, 2, ...$ is a sequence of possibly dependent random variables such that $E(|X_n|) \leq 1$ for all $n$. Suppose that $c_n$ is a sequence of constants that $c_n > 0$ for all $n \geq 1$ and $c_n$ converges to zero as $n \to \infty$. Show that there is a subsequence $n_k$ such that $c_{n_k} X_{n_k}$ converges to zero with probability one.

Philosophy. You can make a living with the Borel-Cantelli Lemmas if you know how to estimate tail probabilities and if you practice up on a few other tricks such as passing to subsequences, making truncations, and completing interpolations.

Further Homework Hints and Requests:

- Staple your homework pages together with a nice stapler. Paper clips tend to pop off. Lost pages are a big mess.
- Be sure to start early on the HWs. You can often pattern a solution on ideas we have used in class, but don't count on your first idea always working. Sometimes the point of the problem is that the natural plan runs into an "objection." You then have to find a way to "over come the objection."

Some Cauchy Coaching

Consider a sequence of real numbers $\{b_j\}$, and consider the two conditions

- $|b_{j+1} - b_j| \leq 1/j$,
- $|b_{j+1} - b_j| \leq 1/j^2$.

Give and example (for yourself, not to hand in, that the first condition does NOT imply that $\{b_j\}$ is Cauchy. Then give a proof (for yourself) that the second condition DOES imply that $\{b_j\}$ is Cauchy. This kind of consideration figures into one of our problems, but it would be too big a hint to say which one.