Statistics 930: Probability Theory — Homework No. 4

INSTRUCTIONS. After you have carefully written a solution to any of the problems, let it sit for a day. After the solution has "ripened," go back and criticize your solution. Finally, write a *de novo* solution that corrects the errors (or slips, or weak statements) in your earlier solution. This process is always worth doing, but it is especially useful when dealing with a problem that is "quantifier rich."

RELATED NOTE: If you have been delaying learning Latex, it's time to bite the bullet. I expect that most solutions to this homework will be done in Latex. For the next homework (No. 5), Latex will be *mandatory.*

We will not rush through the theory around the SLLN but will noodle it this way and that. Accordingly, you may not "know where we are" in Chapter 2 of Durrett. The best plan is for you to both read systematically (i.e. keep reading the successive pages of Durrett) and opportunistically (e.g. read the Wiki to fill small gaps, or dip into related books). It pays to have both organized and disorganized knowledge — *and to know which is which!*

PROBLEM 1. Show that for any random variable X one has

$$\lim_{n\to\infty} E[\cos^{2n}(\pi X)] = P(X \in \mathbb{Z}),$$

where, as usual, $\mathbb{Z}$ denotes the set of integers. What do you get if X is normally distributed?

PROBLEM 2. Show that for any random variable X that is not almost surely equal to zero, we have the limit

$$\lim_{n\to\infty} \frac{E(|X|^{n+1})}{E(|X|^n)} = \sup\{y : P(|X| \geq y) > 0\} = \inf\{y : P(|X| \geq y) = 0\}$$

Hint: Give yourself an "ϵ of room" and consider a limit theorem as having two halves: a limsup half and a liminf half.

PROBLEM 3. We say that a sequence of random variables $\{X_n\}$ converges to X *almost uniformly* if for each $\epsilon > 0$, there exist a set B (B stands for "bad") such that

- $P(B) \leq \epsilon$, and
- X_n converges to X uniformly on the set B^c.

Prove that if X_n converges to X in probability, then there exists a subsequence n_k such that X_{n_k} converges to X almost uniformly as $k \to \infty$. **Note:** There are known theorems in analysis that imply this result, but here you are asked to "argue from first principles" — e.g. by your own construction and invocation of theorems we have proved in class.

PROBLEM 4. Show that if Y is a non-negative random variable on the probability space $(\Omega, \mathcal{F}, P)$ such that $E(|Y|) < \infty$ then there exists a continuous function $f : [0,1] \mapsto \mathbb{R}$ such that $f(0) = 0$ and

$$E(|Y|1_A) \leq f(P(A)) \quad \text{for all } A \in \mathcal{F}.$$

PROBLEM 5. Use the two preceding problems to give a new proof of the *Dominated Convergence Theorem.* That is, prove using these facts that if X_n converges to X in probability and if $|X_n| \leq Y$ and $EY < \infty$ then

$$\lim_{n\to\infty} E(X_n) = E(X).$$

Problem 6. Suppose that X_n, $n = 1, 2, ...$ is a sequence of independent random variables with the standard t-distribution with d degrees of freedom. Prove or disprove the formula

$$\limsup_{n\to\infty} n^{-1/d} X_n = \infty.$$