Statistics 930: Probability Theory — Homework No. 5

Instructions. You want to be sure to keep up with the reading. By the end of the week, we will have covered most of the topics of Chapter 2. We will not do the material on large deviations, but you should give this material a light reading. Instead of large deviations we will do similar work on concentration inequalities, including a basic version of Hoeffding's inequality which has had a big impact on combinatorics and discrete mathematics. We will do the Kolmogorov Zero One Law later. We will also do some of the minor parts of Chapter 2 a little later.

Problem 1. Suppose that ϵ_i, $i =, 2, ...$, is s sequence of independent Gaussian random variables with mean zero and variance σ^2. Suppose that $\rho \in (-1, 1)$ and suppose Y_0 is Gaussian has mean zero and variance given by $\tau^2 = \sigma^2/(1 - \rho^2)$. Suppose also that Y_0 is independent of all the ϵ_i, $i = 1, 2, ...$ Finally, suppose that we have the "evolution equation" given by

$$Y_i = \rho Y_{i-1} + \epsilon_i \quad \text{for } i = 1, 2, ...$$

(1) Show that for all $i \geq 1$, Y_i has mean zero and variance τ^2.
(2) Find a tidy formula for $E(Y_i Y_{i+j})$, $j = 1, 2, ...$ As a check, your formula should be of the form $f(j)$ for some very simple f.
(3) Let S_n be the partial sum of the Y_i. Show that $\text{Var} S_n = O(n)$.
(4) State and prove an appropriate WLLN for S_n.
(5) Can you prove a SSLN?

Hint: For the last piece, think of the techniques have you seen that might help. It often pays to consider the simplest methods first.

Problem 2. Given a probability space $(\Omega, \mathcal{F}, P)$ and random variables X and Y define $\rho(X, Y) = E(\min(|X - Y|, 1))$.

(1) Show that ρ is a metric on the space of all random variables on $(\Omega, \mathcal{F}, P)$.
(2) Show that ρ "metrizes the topology of convergence in probability;" that is, show that X_n converges to X in probability if and only if $\rho(X_n, X) \to 0$ as $n \to \infty$.
(3) Show that "convergence in probability is preserved by continuous mappings"; that is, show that if X_n converges to X in probability then for any continuous function, $f(X_n)$ converges to $f(X)$ in probability.
(4) Show that the corresponding statement is false for convergence in L^1. That is, give examples of random variables X_n and X and a continuous function f such that $E|X_n - X|$ converges to zero but $E|f(X_n) - f(X)|$ does not converge to zero.

Note: One of the required properties of a metric space is that $\rho(X, Y) = 0$ implies that $X = Y$. Here the equality stands for "almost surely equal" or equal on a set of probability one. In general, we identify a random variable with the equivalency class of all random variables Y such that $P(X = Y) = 1$.

Problem 3. Suppose that X_i are IID with the uniform distribution on $[0, 1]$. Let

$$N = \min\{k : X_k \leq X_{k+1}\}.$$

Find $P(N > n)$ and EN.
Hint: I would not assign such a problem if the answer were not beautiful. It also contrasts nicely with our "salary exercise" from an earlier HW. Be careful when you calculate EN, i.e. make sure your formulas are right.

PROBLEM 4. Suppose X_i $i = 1, 2...$ are IID, $E|X_i| < \infty$ and $EX_i = 0$. Let

$$S_n = X_1X_2 + X_2X_3 + \cdots + X_{n-1}X_n.$$

Prove the SLLN: S_n/n converges to 0 with probability one.
HINT: One always has the choice of using past results or using a variation on the proofs of past results. Here, as usual, both methods work; here, as usual, one method is much quicker than the other.

PROBLEM 5. Study the proof of the SLLN that follows the Kolmogorov plan and study the proof that follows the plan in the *Monthly* article. When you understand the details of both proofs, answer the following questions:

(1) Both used truncation; but they were of different kinds. The truncations were even applied to different kinds of random variables. Describe these differences in words, as if you were explaining them to a friend on bus.
(2) Both proofs used maximal inequalities. How are these the inequalities different and how does this difference show up in the proof.
(3) We used interpolation in some of our SLLN proofs. What were those theorems? Why did we not need an interpolation in the Kolmogorov proof and in the class note proof? What took the place of interpolation?
(4) We needed to use Kronecker's lemma in the Kolmogorov proof, but it is not used in the proof that uses the L^1 maximal inequality. What accounts for this difference?

FINAL COMMENTS.

There is a third proof of the adult strength SLLN given in the text. This is Etemadi's argument, and you should read it. The calculations are very close to those we did when we followed the Kolmogorov plan, but the logic is a little different. I also give a link to one of Tao's blog articles on the SLLN. He approaches the SLLN along the lines of the Estimated plan but he deals with the details in a way that makes the computations more transparent, if you are comfortable with O and o notations. It is thought provoking to read the two treatments, and, if you can possibly spare the time, I encourage you to engage this extra reading with all of the energy you can muster.