Statistics 930: Probability Theory — Homework No. 7

INSTRUCTIONS. You should complete the reading of the text through Section 3.5. You should have a clear understanding of Levy's inversion formula and Levy's continuity theorem. This is a shorter assignment since we had a test on Monday.

PROBLEM 1.

Suppose that for all $x \geq 0$, and suppose the random variable X satisfies the bound

$$P(X \geq A + B\sqrt{x}) \leq e^{-x},$$

where A and B are nonnegative constants and n is an integer. Show that one has

$$E[X] \leq A + \frac{B\sqrt{\pi}}{2}.$$

Hint: It may be useful to note that $X \leq A + (X - A)_+$, and you may need to recall the value of $\Gamma(1/2)$.

PROBLEM 2.

- Remember, compute, or look up the characteristic function of the Gamma density. Suppose that X_1 and X_2 are independent and identically distributed and suppose that $X_1 + X_2$ has the exponential distribution with mean one. Write down an explicit formula for the density of X_1. Marvel at the beauty of this.
- If Y_1 and Y_2 are independent, normal, mean zero, variance one, find the distribution of
$$R = \sqrt{Y_1^2 + Y_2^2}$$

PROBLEM 3.

Show that there does not exist a characteristic function $\phi(t)$ such that the derivative $\phi'(t)$ exists for all t and such that $\phi'(t)$ is also a characteristic function. Note: This would be super easy if we were to assume that the distribution associated with $\phi(t)$ had a finite first moment. Unfortunately we know that it is possible for $\phi'(t)$ to exist with out the first moment existing. You need an argument where you do not assume that you have a first moment.

PROBLEM 4. Consider the sum

$$A_n(t) = \frac{1}{2n+1} \sum_{k=-n}^{n} e^{itk/n}.$$

- This is the characteristic function of a random variable Z_n. Describe Z_n as the result of an experiment.
- For large n the discrete random variable Z_n looks a lot like a random variable that has a density. Describe that density.
- Now, using the theory of characteristic functions, explain why it is obvious that one has
$$\lim_{n \to \infty} \frac{1}{2n+1} \sum_{k=-n}^{n} e^{itk/n} = \frac{\sin t}{t}.$$
- Now explain why the last formula is actually obvious from the definition of the Riemann integral.

A useful lesson on from this exercise is that the integral of the RHS over $(-\infty, \infty)$ is π but the integral of the LHS over this interval is (pretty much) non-sense. This observation gives us a reminder that it is fine to **calculate boldly**, but one still has to avoid drifting off into silliness.