Statistics 530: Probability Theory — Homework No. 7

INSTRUCTIONS. You should complete the reading of the text through Section 3.5. You should have a clear understanding of Levy's inversion formula and Levy's continuity theorem. These result should be part of your active, short-term memory — suitable for inclusion on a No-Name Quiz.

You should also master the proofs of these fundamental theorems; at a minimum you need to have a clear idea of the *tools* that we developed to prove these results. In particular, you should review the diagonal argument and the notion of sequential compactness. The HW is a little easy this week to give you time for reading and review.

PROBLEM 1.A WARM-UP.

Suppose that for all $x \geq 0$, and suppose the random variable X satisfies the bound

$$P(X \geq A + B\sqrt{x}) \leq e^{-x},$$

where A and B are nonnegative constants and n is an integer. Show that one has

$$E[X] \leq A + \frac{B\sqrt{\pi}}{2}.$$

Hint: It may be useful to note that $X \leq A + (X - A)_+$, and you may need to recall the value of $\Gamma(1/2)$.

PROBLEM 2. Some quick shots:

- Remember, compute, or look up the characteristic function of the Gamma density. Suppose that X_1 and X_2 are independent and identically distributed and suppose that $X_1 + X_2$ has the exponential distribution with mean one. Write down an explicit formula for the density of X_1. Marvel at the beauty of this.
- If Y_1 and Y_2 are independent, normal, mean zero, variance one, find the distribution of

$$R = \sqrt{Y_1^2 + Y_2^2}$$

PROBLEM 3.

(a) Is there a characteristic function $\phi(t)$ such that the derivative $\phi'(t)$ exists for all t and such that $\phi'(t)$ is also a characteristic function? Either construct such a $\phi(t)$ or prove that it does not exist.

(b) Give an example of a pair of characteristic functions ϕ_1 and ϕ_2 such that $\phi_1(t) = \phi_2(t)$ for all $|t| \leq 1$ but $\phi_1(t) \neq \phi_2(t)$ for all $t \notin [-1, 1]$.

PROBLEM 4. Let $\mathbf{v}$ be a unit vector in $\mathbb{R}^n$ and let $\mathbf{x}$ be chosen according to the uniform distribution on $\{-1, 1\}^n$. Show that

$$3/16 \leq P(|\mathbf{v} \cdot \mathbf{x}| \geq 1/2).$$

Note that this bound does not depend on the dimension n.

Hint: This is a lower bound problem, so a certain favorite tool springs to mind. This favorite tool speaks about a nonnegative random variable. Here the obvious choice is $|\mathbf{v} \cdot \mathbf{x}|$, but the obvious choice runs into trouble. What is the second most obvious choice? Show that this choice works.

PROBLEM 5. Consider the sum

$$A_n(t) = \frac{1}{2n+1} \sum_{k=-n}^{n} e^{itk/n}.$$

- This is the characteristic function of a random variable Z_n. Describe Z_n as the result of an experiment.
- For large n the discrete random variable Z_n looks a lot like a random variable that has a density. Describe that density.
- Now, using the theory of characteristic functions, explain why it is obvious that one has
$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} e^{itk/n} = \frac{\sin t}{t}.$$
- Now, to be unfancy, explain why the last formula is obvious from the definition of the Riemann integral.
- Just for fun, note that the integral of the RHS over $(-\infty, \infty)$ is π but the integral of the LHS over this interval is (pretty much) non-sense. This is a reminder that it is fine to **calculate boldly**, but you can't just drift off into silliness.